RUSSIAN SPACE ENGINEERING WORKSHOP

Thermally Loaded Structures

July 20 - 21, 1993 • The University of Alabama in Huntsville

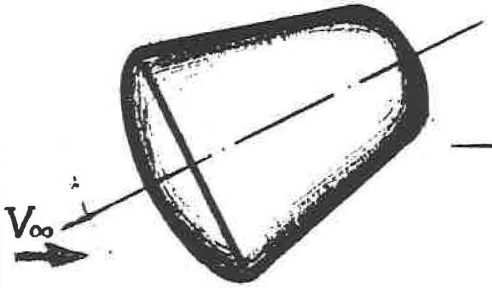
Course Notebook Contents

I.

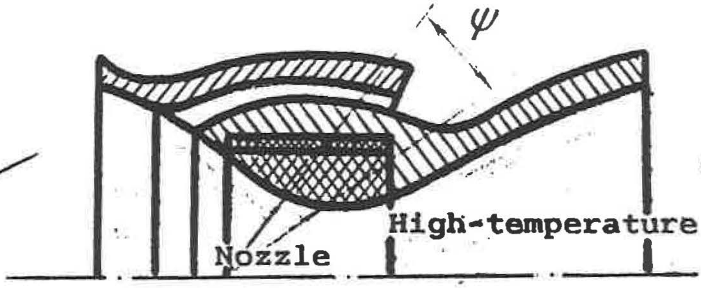
Introduction to Thermal Design and Testing

Heat - loaded Vehicles and Structures (Examples)

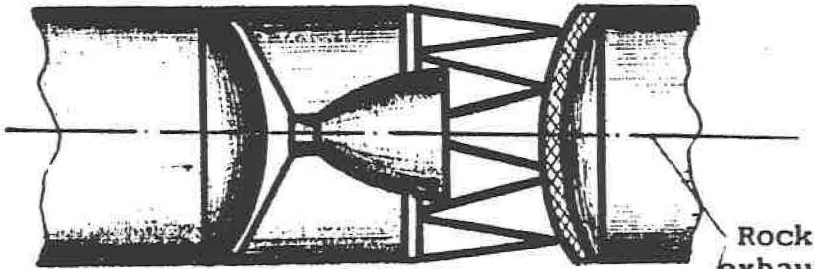
Shock-wave



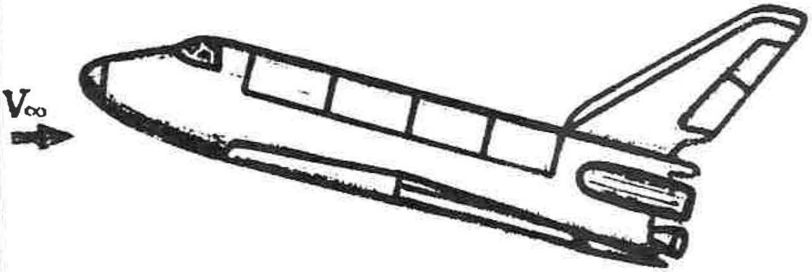
Re-entry Vehicle



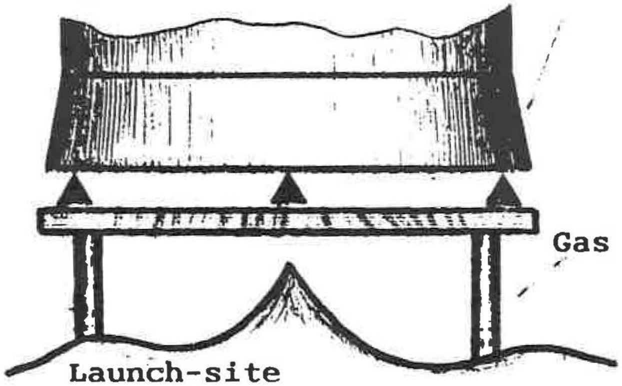
Gas Rudder



Stage Separation Unit



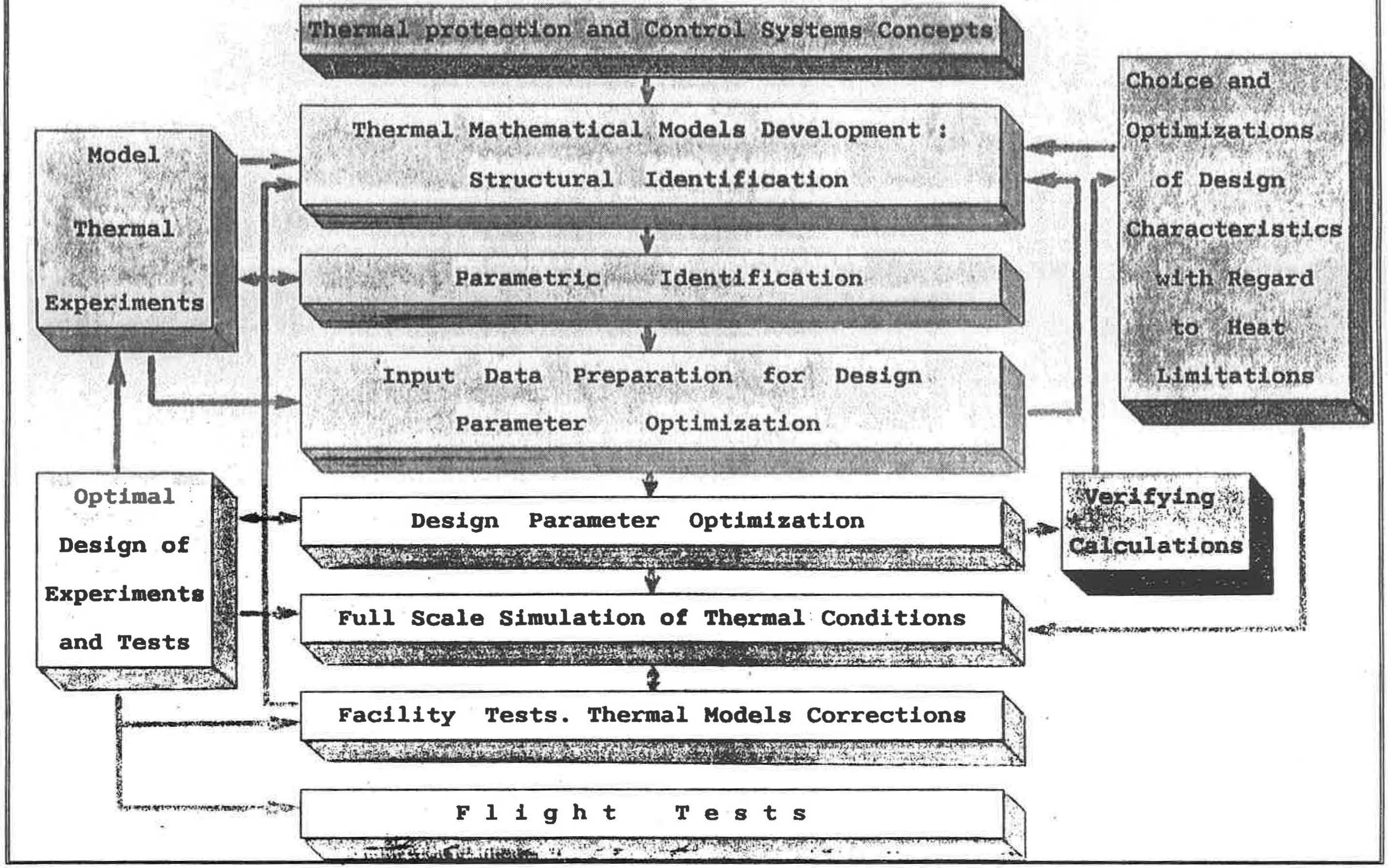
Aerospace craft



Launch-site

TP 1

Block Diagram of Thermal Design and Testing



Thermal protection and Control Systems Concepts

Thermal Mathematical Models Development : Structural Identification

Parametric Identification

Input Data Preparation for Design Parameter Optimization

Design Parameter Optimization

Full Scale Simulation of Thermal Conditions

Facility Tests. Thermal Models Corrections

Flight Tests

Model Thermal Experiments

Optimal Design of Experiments and Tests

Choice and Optimizations of Design Characteristics with Regard to Heat Limitations

Verifying Calculations

Optimal Thermal Design (OTD)

The main purpose of OTD is a determination of the design solutions and parameters of the thermal protection and control.

General statement of corresponding optimization problem:

$\vec{p}$ [illegible] $: \min\limits_{\vec{p} \in G} J[\vec{p}],$

$$G = \{\vec{p} : g_i(\vec{p}, T(\vec{x}, \tau, \vec{p})) \leq 0, \quad i = 1, 2, \ldots, \ell\},$$

$$T(\vec{x}, \tau, \vec{p}) = A[T(\vec{x}, \tau, \vec{p}), \vec{p}]; \quad \vec{p} \in R^K,$$

where $\vec{p}$ is a vector of design parameters;

G is a set of admissible solutions;

$\{g_i\}_1^{\ell}$ are physical and technical restrictions;

J is a criterion function;

T is temperature;

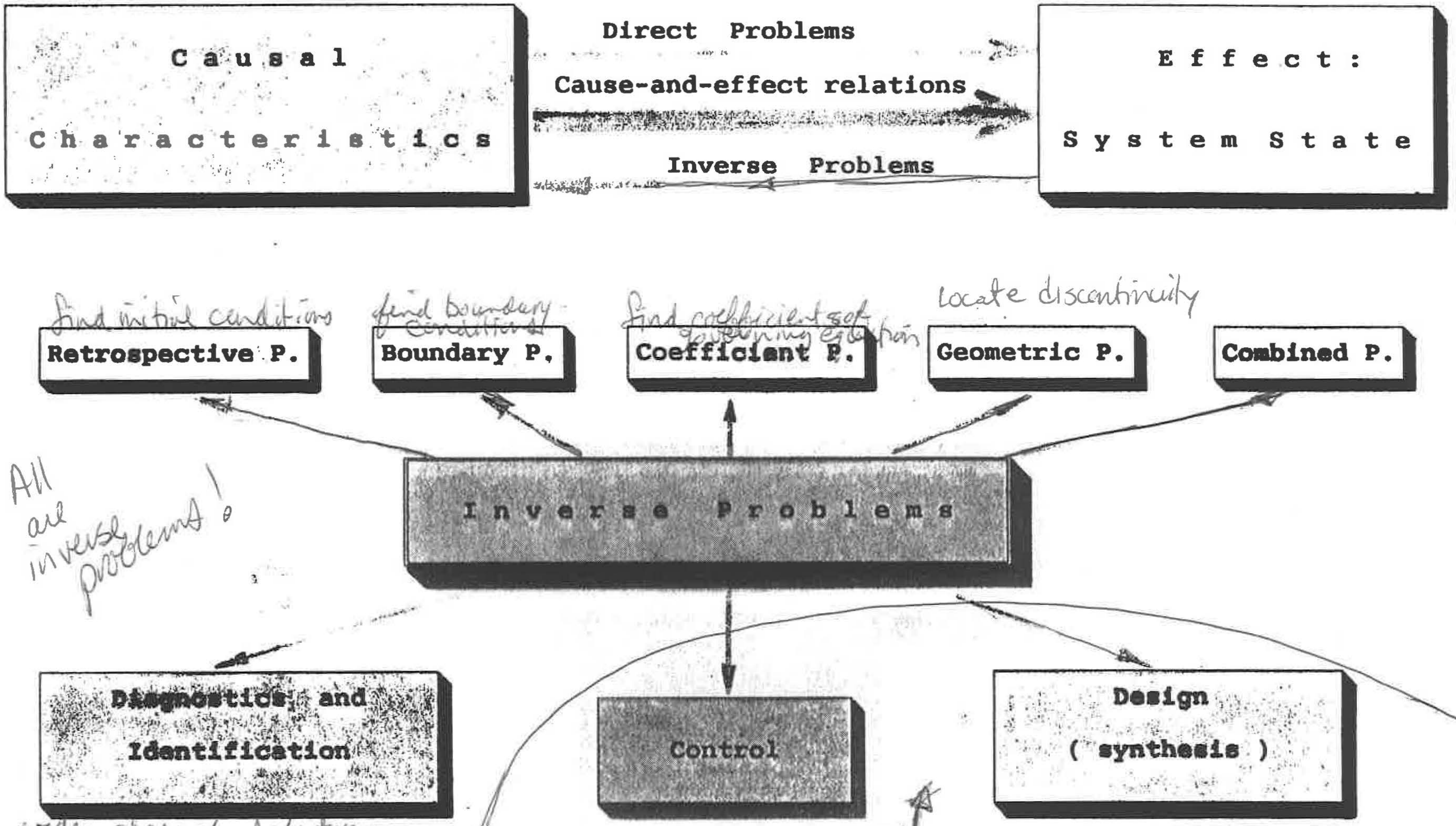
τ is time;

$\vec{x}$ is a vector of spatial variables;

A is an operator of thermal mathematical model of system under design

II.
Direct and Inverse Problems

Direct and Inverse Problems



Causal Characteristics

Direct Problems

Cause-and-effect relations

Inverse Problems

Effect: System State

find initial conditions

find boundary conditions

find coefficient of governing equation

locate discontinuity

Retrospective P.

Boundary P.

Coefficient P.

Geometric P.

Combined P.

All are inverse problems!

Inverse Problems

Diagnostics and Identification

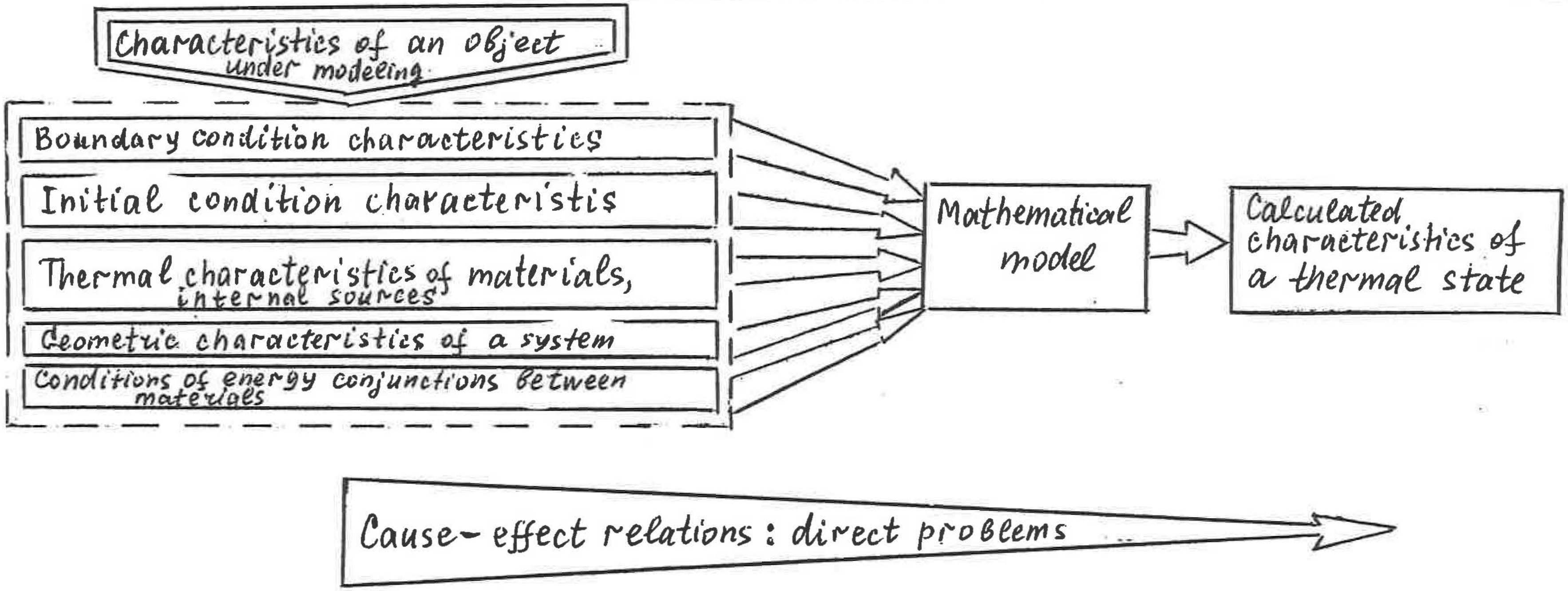
Control

Design (synthesis)

wider class of solution, the larger the errors that can arise.

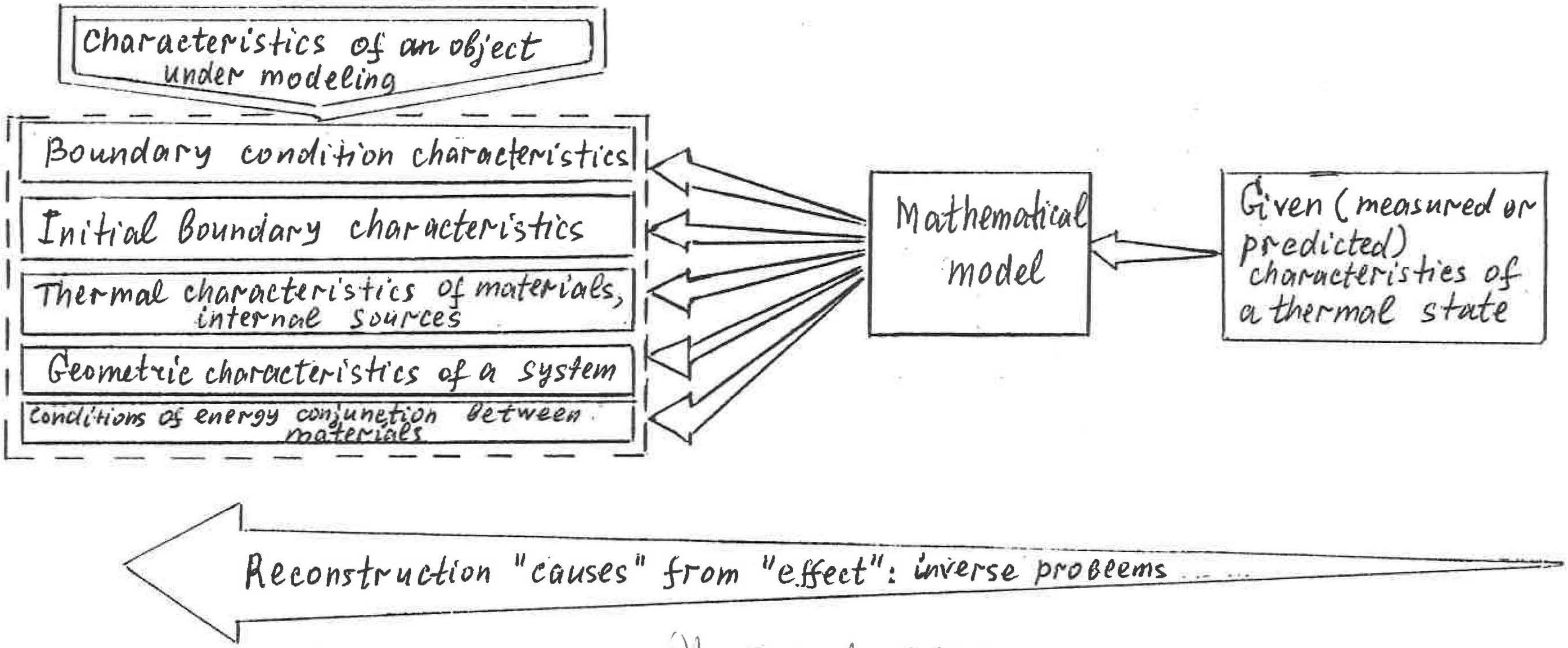
Fundamentally different

Direct Problems in Mathematical Modeling



Direct problems are analysis problems

Inverse Problems in Mathematical Modeling



Characteristics of an object under modeling

Boundary condition characteristics

Initial Boundary characteristics

Thermal characteristics of materials, internal sources

Geometric characteristics of a system

Conditions of energy conjunction between materials

Mathematical model

Given (measured or predicted) characteristics of a thermal state

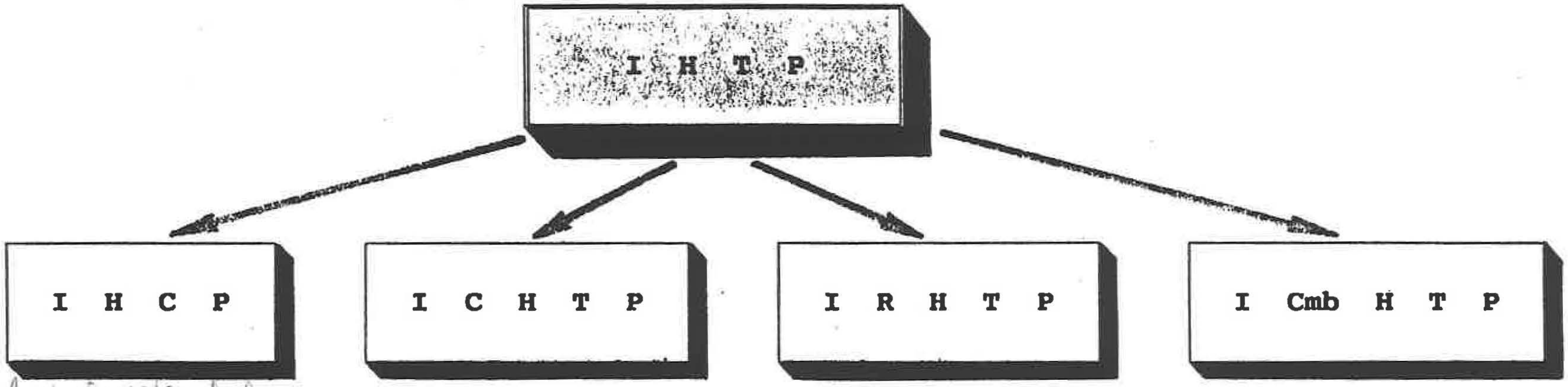
Reconstruction "causes" from "effect": inverse problems ... ...

ill-posed-ness

Inverse problems are design, diagnostics and identification problems

Inverse Heat Transfer Problems

Another classification key - kinds of IHTP



IHTP

IHCP

ICHTP

IRHTP

ICmbHTP

best investigated
most widely used

IHTP - Inverse Heat Transfer Problems

IHCP - Inverse Heat Conduction Problems

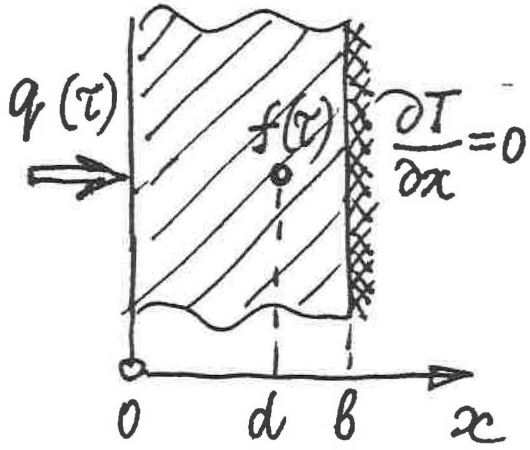
ICHTP - Inverse Convective Heat Transfer Problems

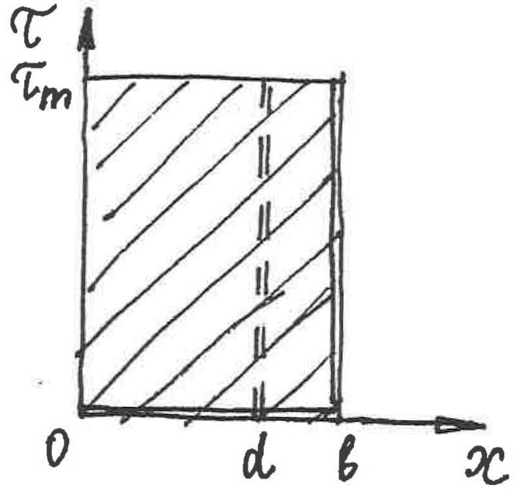
IRHTP - Inverse Radiation Heat Transfer Problems

ICmbHTP - Inverse Combined Heat Transfer Problems

III.
Statements of Inverse Heat Transfer Problems

Boundary IHCP
(an example)





Mathematical Statement of this Problem

Heat - conduction equation :

$$C(T)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(k(T)\frac{\partial T}{\partial x}\right), \quad \begin{matrix} 0 < x < b \\ 0 < \tau \leqslant \tau_m \end{matrix}$$

Initial temperature distribution :

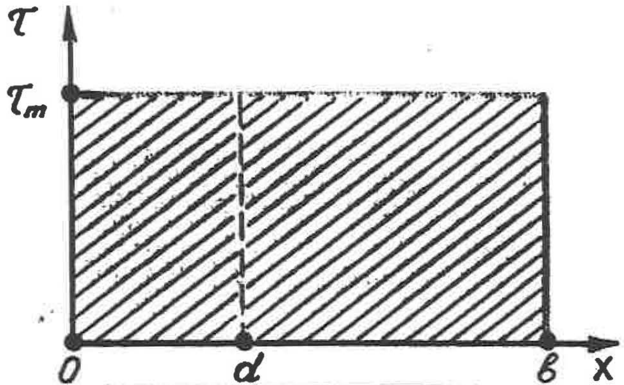
$$T(x,0) = \varphi(x), \qquad 0 \leqslant x \leqslant b$$

Internal temperature history :

$$T(d,\tau) = f(\tau), \; 0 \leqslant \tau \leqslant \tau_m, \; 0 < d \leqslant b$$

Boundary condition:

$$\frac{\partial T(b,\tau)}{\partial x} = 0, \qquad 0 \leqslant \tau \leqslant \tau_m$$



Unknown functions :

$$\boxed{q\;(\tau)} = -k\left.\frac{\partial T}{\partial x}\right|_{x=0},$$

$$\boxed{T_w(\tau)} = T(0,\tau)$$

Heat balance equation :

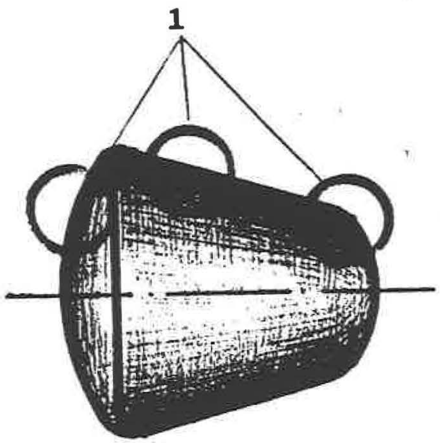
$$\boxed{q_\Sigma(\tau)} = q\;(\tau) + \overset{\overset{\text{emissivity}}{\uparrow}}{\varepsilon}\sigma T_w^4(\tau)$$

Heat transfer coefficient :

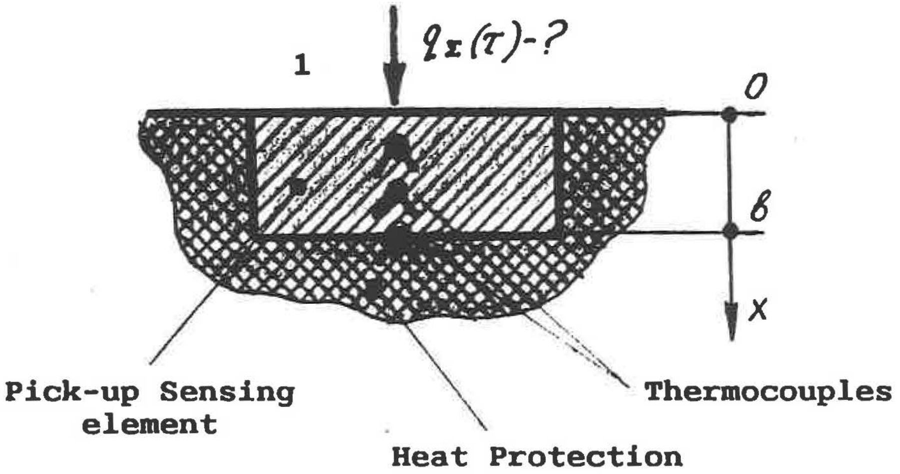
$$\boxed{h(\tau)} = \frac{q_\Sigma(\tau)}{T_\delta(\tau) - T_w(\tau)}$$

What are IHTP for? (Three examples)

* **Example 1:** Diagnostics of transient heat loads acting on the heat-stressed structures of flight vehicles and launching facilities

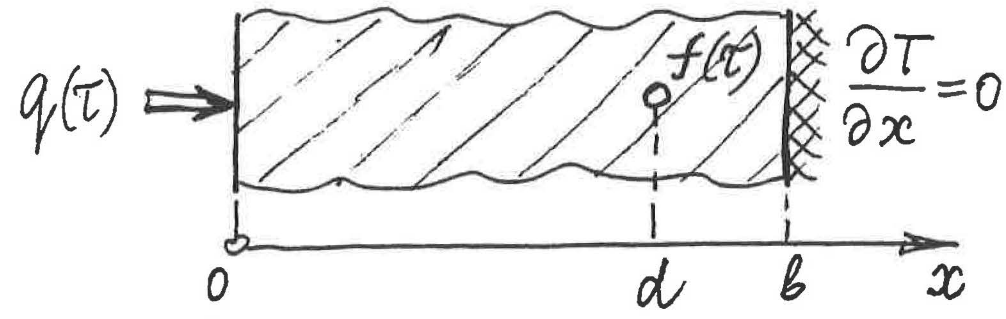


Re-entry vehicle

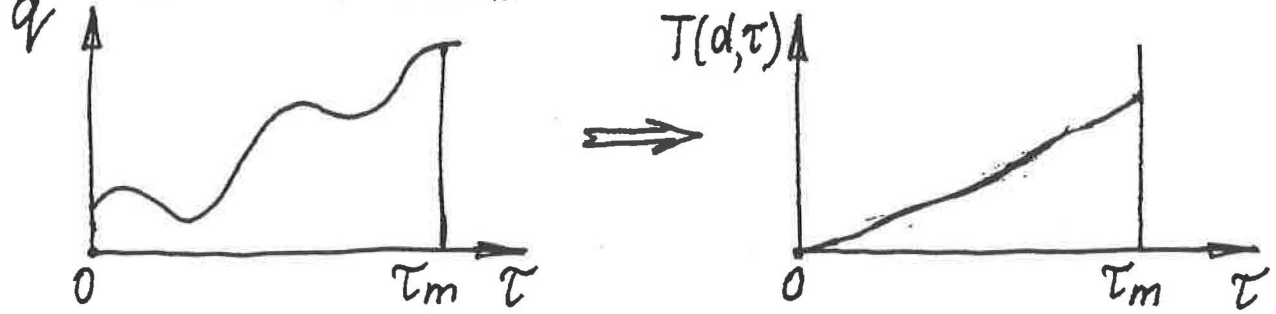


A lay - out of the sensor of unsteady heat fluxes

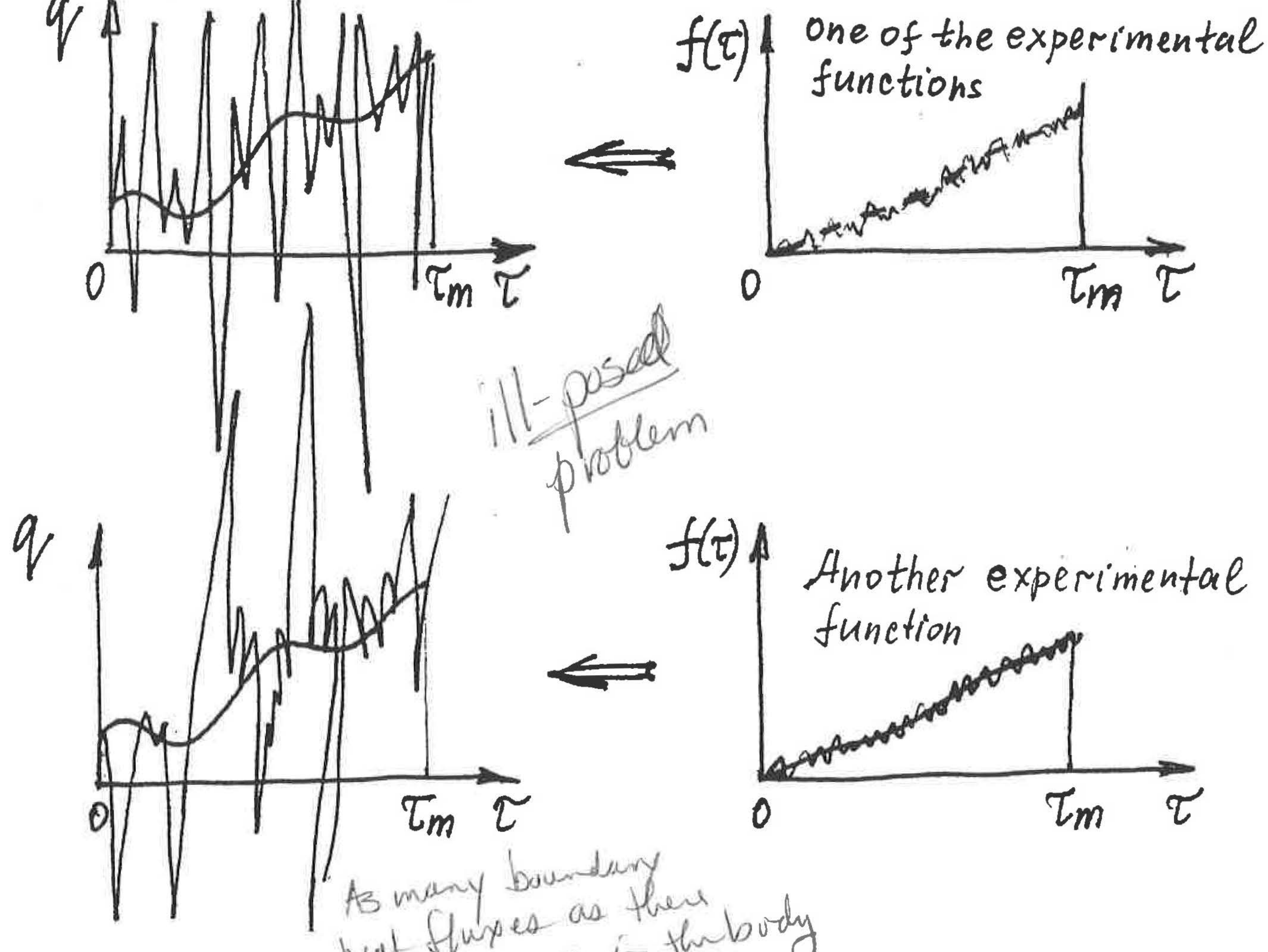
Physical explanation of ill-posedness of IHCi



Direct Problem



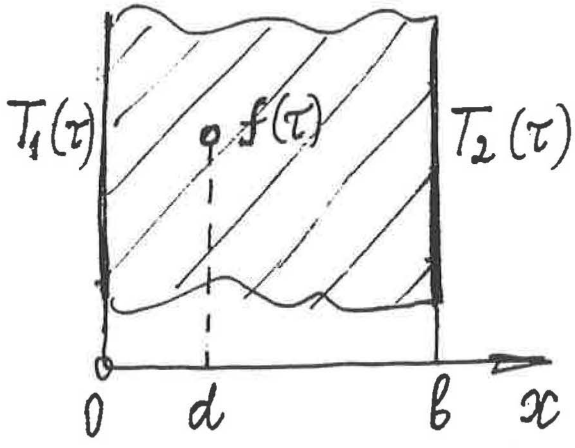
Inverse Problem

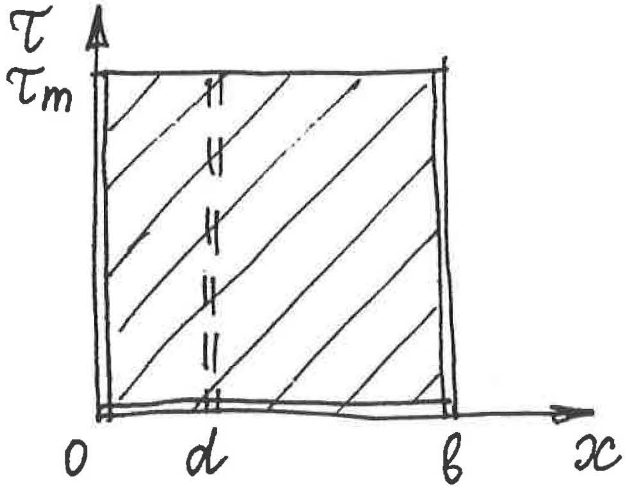


ill-posed problem

As many boundary heat fluxes as there are sensors in the body

Coefficient IHCP (a example)





$$C(T)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right), \quad 0<x<b, \quad 0<\tau\le\tau_m$$

$$T(x,0) = \varphi(x), \quad 0\le x\le b$$

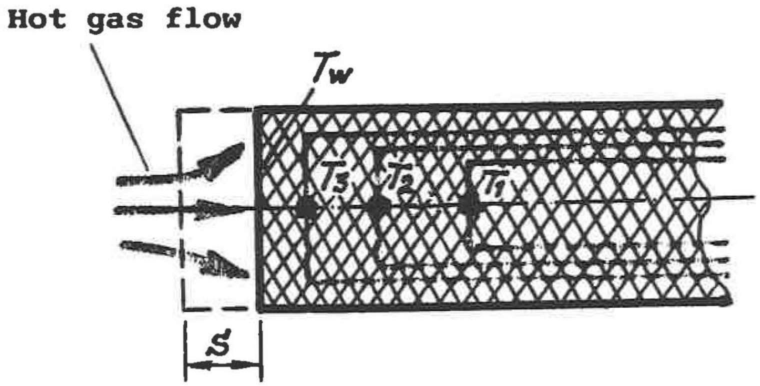
$$\left.\begin{array}{l} T(0,\tau) = T_1(\tau) \\ T(b,\tau) = T_2(\tau) \\ T(d,\tau) = f(\tau), \end{array}\right\} \quad 0\le\tau\le\tau_m$$

$0<d<b$

Unknown function : $\lambda = \lambda(T), T_{min}\le T\le T_{max}$

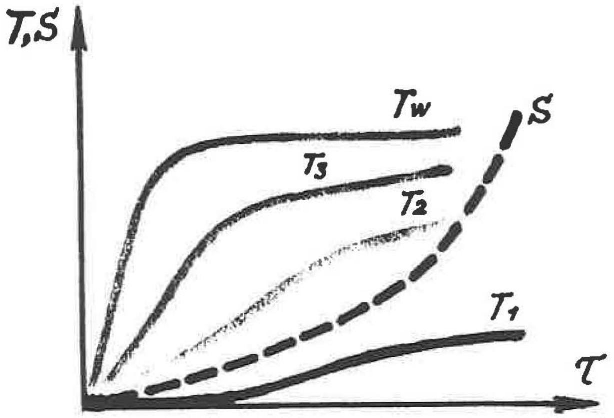
Can estimate as many coefficients as we have sensors

Example 2: Simulation of the required conditions of specimen heating on special test facilities (plasmatrons, jets of rocket engines and others) with subsequent processing of temperature measurements by methods of inverse problems.



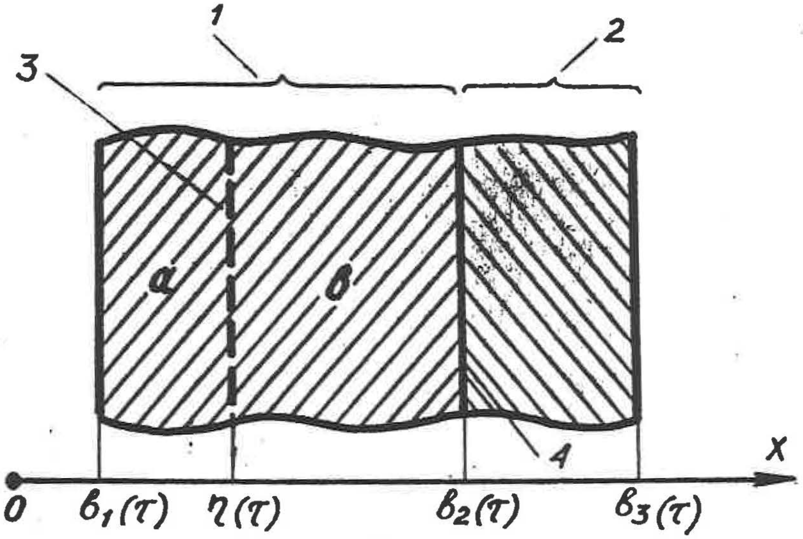
A specimen of the material with thermocouples

(S is a material ablation layer at a moment τ)

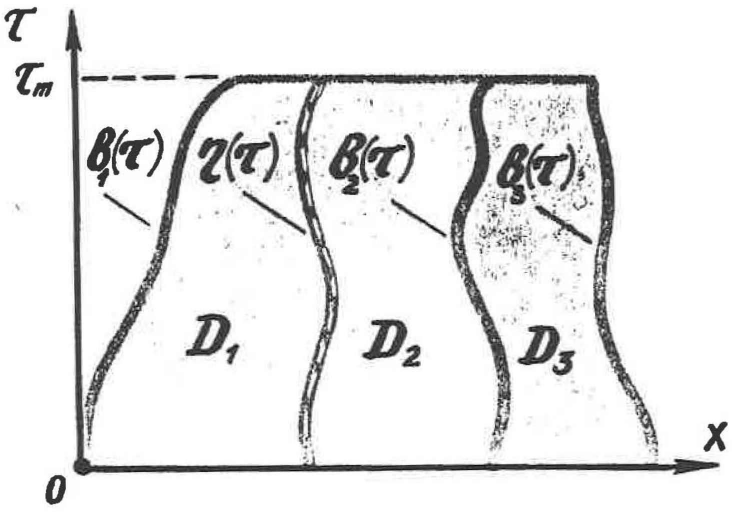


A Thermogram

Onedimensional Statement



A two-layered plate with phase transition: or ablation



A domain with mobile boundaries

1 , 2 - the first and the second layers;

3 - boundary of phase transition;

4 - the line of contact between layers;

a , b - two phase states of material of the first layer

Mathematical Formulation

For porous materials

$$\boxed{C_j \frac{\partial T_j}{\partial \tau} = \frac{\partial}{\partial x}\left(k_j \frac{\partial T_j}{\partial x}\right) + P_j \frac{\partial T_j}{\partial x} + S_j , \qquad (x,\tau) \in D_j(\tau), \quad j = 1,2,3}$$

convective term

distributed heat source

"generalized" heat conduction equation

Conjuction conditions: at interface surfaces

$$T_1(\eta(\tau)-0,\tau) = T_2(\eta(\tau)+0,\tau), \qquad k_1 \left.\frac{\partial T_1}{\partial x}\right|_{x=\eta(\tau)-0} - k_2 \left.\frac{\partial T_2}{\partial x}\right|_{x=\eta(\tau)+0} = r\frac{\partial \eta}{\partial \tau}$$

$$T_2(b_2(\tau)-0,\tau) = T_3(b_2(\tau)+0,\tau) - R k_2 \left.\frac{\partial T_2}{\partial x}\right|_{x=b_2(\tau)-0}$$

$$k_2 \left.\frac{\partial T_2}{\partial x}\right|_{x=b_2(\tau)-0} = k_3 \left.\frac{\partial T_3}{\partial x}\right|_{x=b_2(\tau)+0}$$

contact resistance between layers

r volumetric heat of melting

Initial conditions:

$$T_j(x,0) = \xi_j(x), \qquad j = 1,2,3$$

$$j=1:\ b_1(0) \le x \le \eta(0); \qquad j=2:\ \eta(0) \le x \le b_2(0); \qquad j=3:\ b_2(0) \le x \le b_3(0)$$

Possible boundary conditions:

I. $$T_j(B_j(\tau),\tau) = t_j(\tau), \quad j = 1,3$$

II. $$-k_j \left.\frac{\partial T_j}{\partial x}\right|_{x=B_j(\tau)} = q_j(\tau), \quad j = 1,3$$

III. **Newton conditions** convective

$$-k_j \left.\frac{\partial T_j}{\partial x}\right|_{x=B_j(\tau)} = \overset{\text{heat transfer coeff.}}{h_j}\left[T_j(B_j(\tau),\tau) - \overset{\text{ambient temp}}{T_j^*}(\tau)\right], \quad j = 1,3$$

IV. $$-k_j \left.\frac{\partial T_j}{\partial x}\right|_{x=B_j(\tau)} = \overset{\text{convection}}{h_j\left[T_j(B_j(\tau),\tau) - T_j^*(\tau)\right]} + \overset{\text{absorptivity}}{A_j}\, \underset{\text{radiative heat flux}}{q_{rj}} -$$

$$-\underset{\text{radiation}}{\varepsilon_j \sigma T_j^4(B_j(\tau),\tau)} + \underset{\text{heat source}}{g_j}, \quad j = 1,3$$

- Volumetric heat capacities C_j
- Thermal conductivity k_j
- Convection coefficients P_j
- Sources S_j
- Movement of boundaries B_1, B_2, B_3 and of phase transfer front η
- Volumetric heat of phase transfer τ
- Thermal contact resistance R
- Initial temperatures $\xi_j(x)$
- Boundary temperatures t_j
- Heat fluxes q_j
- Heat transfer coefficients h_j
- Environmental temperatures T_j^*
- Absorptivity A_j
- Emissivities ε_j
- Surface heat sources g_j

(CONTINUATION)

Additional conditions:

$$T(d_i,\tau) = f_i(\tau), \quad i = \overline{1,N}, \quad d_i \in \bar{D}_1 \cup \bar{D}_2 \cup \bar{D}_3$$

$$d_i = const_i \quad \text{or} \quad d_i = d_i(\tau)$$

IHCP: $T(x,\tau)$ + some causal characteristics - ?

Kinds of IHCP:

"Reverse Time"

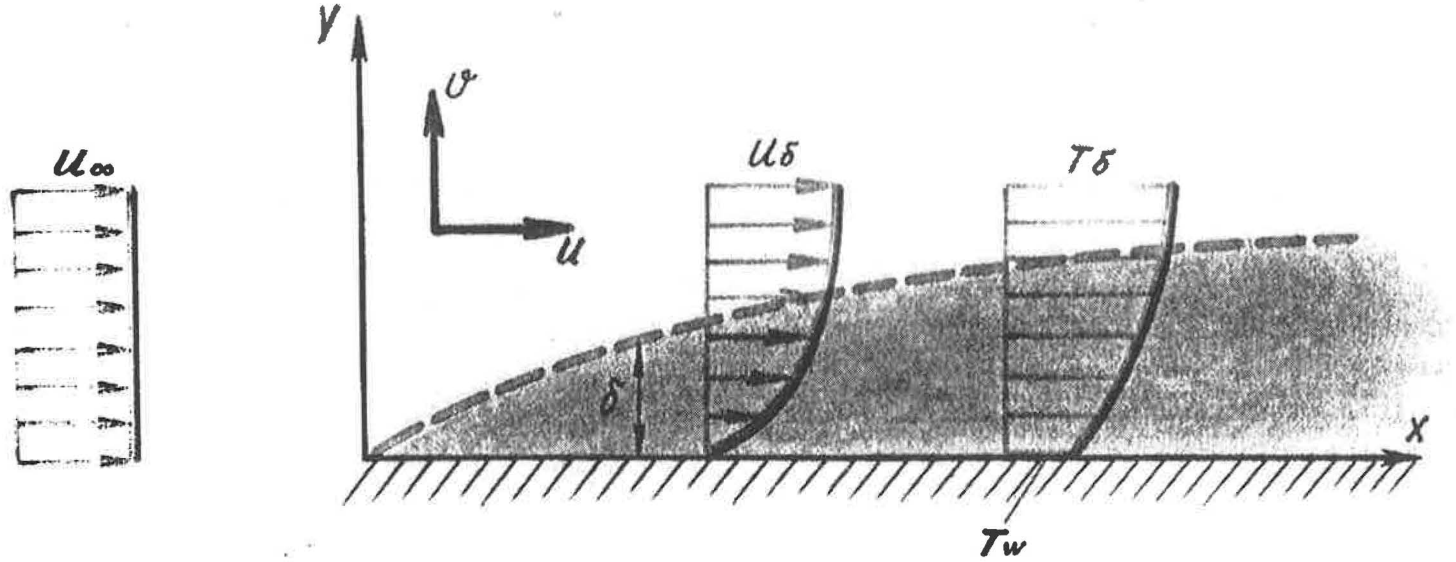
1. **Retrospective Heat Conduction Problem** : $\{\xi_j(x)\}_{j=1,2,3} = ?$
2. **Boundary IHCP** : $\{t_j(\tau), q_j(\tau), T_j^*(\tau), A_j(T_{w_j},\tau), \varepsilon_j(T_{w_j},\tau), g_j(T_{w_j},\tau)\}_{j=1,3} = ?$
3. **Coefficient IHCP** :

$$\{C_j(x,\tau,T_j(x,\tau)), k_j(x,\tau,T_j(x,\tau)), P_j(x,\tau,T_j(x,\tau)), S_j(x,\tau,T_j(x,\tau))\}_{j=1,2,3} = ?$$

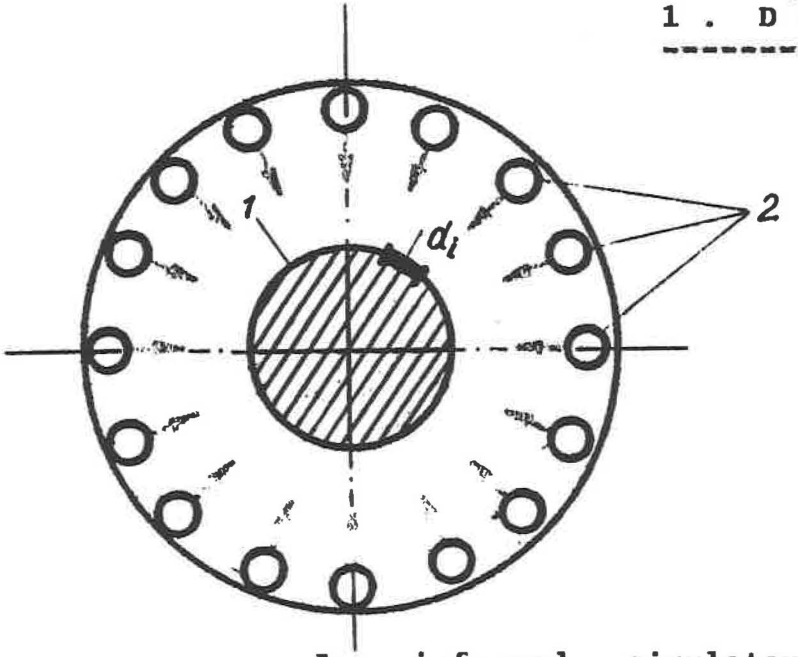
4. **Geometric IHCP** : $\{b_j(\tau), \eta(\tau)\}_{j=1,2,3} = ?$

one or more ← 5. **Combined IHCP**

Inverse Convective Heat Transfer Problems



1. Reconstruction of $T(y)$
2. Estimation of turbulent viscosity μ_T and thermal conductivity k_T
3. Determination of catalytic activity of a solid wall



1. Discrete form

Heat balance equations :

$$\pi \sum_{j=1}^{m} J_j \sum_{\kappa=1}^{n} \varphi_{d_i - j_\kappa} A_{i-j} = q_i \, , \quad i = \overline{1,m} \, ,$$

where $\varphi_{d_i - j_\kappa}$ is a local angular coefficient ,

A_{i-j} is an absorptivity ,

$\boxed{q_i \text{ is a known heat flow}}$

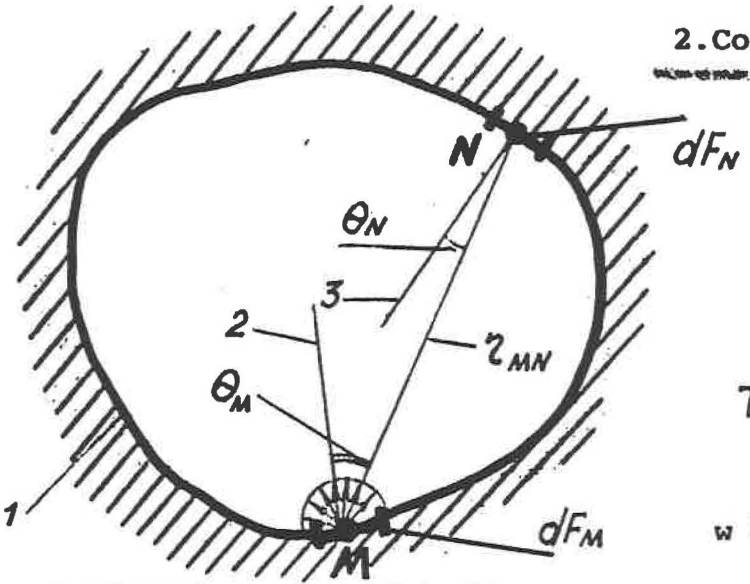
An infrared simulator :

1 - test object

2 - radiative heaters

Radiation intensities of modules J_j - ?

Inverse Radiation Heat Transfer Problems



A closed system of bodies :

1 - diffusively radiating surface F ;

2,3 - normals to the surface at points M and N , respectively

2.Continuous form

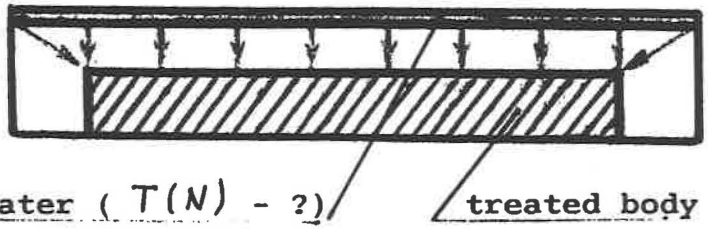
$$\int_F E_{eff}(N)\,\frac{\cos\theta_M\cos\theta_N}{\pi r_{MN}^2}\,dF_N = E_{in}(M),\quad M\in F$$

Temperature fild :

$$T(N)=\left\{\frac{1}{\sigma\varepsilon}\left[E_{eff}-(1-A)E_{in}\right]\right\}^{1/4},\quad N\in F,$$

the unknown function

where E_{eff} , E_{in} are half-sperical densities of effective and incident radiation respectively

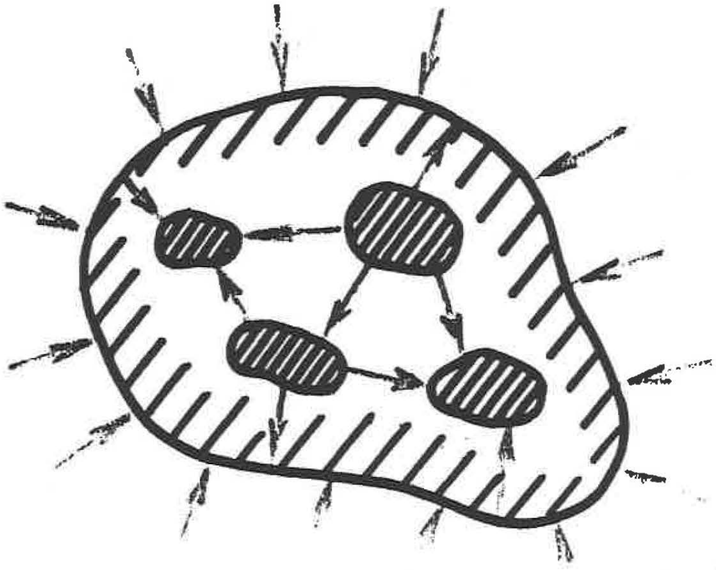


Temperature distribution on body surfaces $T(M)$ - ?

heater ($T(N)$ - ?) treated body ($T(M)$, $q_{res}(M)$ are given distributions)

stationary and linear problems

1 . Inverse problem of heat transfer in engineering systems



A system of bodies
(the heat fluxes are arbitrary shown by arrows)

Heat balance equations : ℓ, j

$$C_\ell \frac{dT_\ell}{d\tau} = \sum_{j=1}^{n} a_{\ell j}(T_j - T_\ell) + \sigma \sum_{j=1}^{n} b_{\ell j}(T_j^4 - T_\ell^4) + Q_\ell$$

$$T_\ell(0) = T_0 , \quad \ell = \overline{1,n} , \quad 0 < \tau \leq \tau_m$$

Direct measurements :

$$T_\ell(\tau) = f_\ell(\tau) , \quad \ell = \overline{1,\kappa} , \quad \kappa \leq n$$

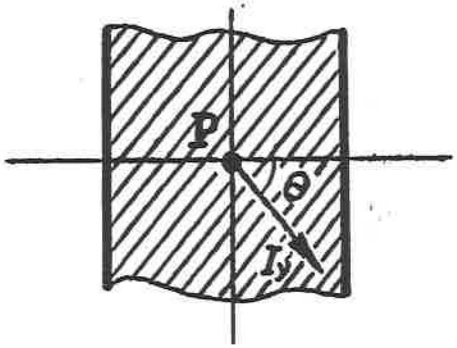
Causal characteristics :

- * C_ℓ is a heat capacity of node ℓ
- * $a_{\ell j}$ is a conductive and convective heat exchange coefficient between nodes ℓ and j

conduction or convection or both

- * $b_{\ell j}$ is a radiative heat exchange coefficient between nodes ℓ and j
- * Q_ℓ is a heat input from space to node and internal generated heat power within node ℓ

2. Inverse problem of conductive-radiative heat transfer



A semi - transparent body

$$C\frac{\partial T}{\partial \tau} = div\left(k\, grad\, T\right) - div\,\vec{q}_r\,,$$

$$q_r = \int_0^\infty \int \vec{\ell}\, I_\nu \, d\ell \, d\nu\,,$$

spectrum of radiant energy

$$\vec{\ell}\, grad\, I_\nu = (K_\nu + \beta_\nu)\left[-I_\nu(\vec{\ell},P) + (1 - Sc_\nu)\, B_\nu(P) + \right.$$

$$\left. + \frac{Sc_\nu}{4\pi}\int_{4\pi} I_\nu(\vec{\ell},P)\,\theta_\nu(\vec{\ell},\vec{\ell}')\, d\omega\right. -$$

Schuster number (like Nusselt number in convection)

– equation of radiation transport ,

where :

$\vec{q}_r$ is a radiation energy flow vector; I_ν is a radiation spectral intensity;

$\vec{\ell}$ is a single direction vector of radiation propagation in point P ;

K_ν , β_ν are coefficients of absorption and dispersion of radiation with the frequency ν ; $Sc_\nu = \beta_\nu / (K_\nu + \beta_\nu)$ is a Schuster number;

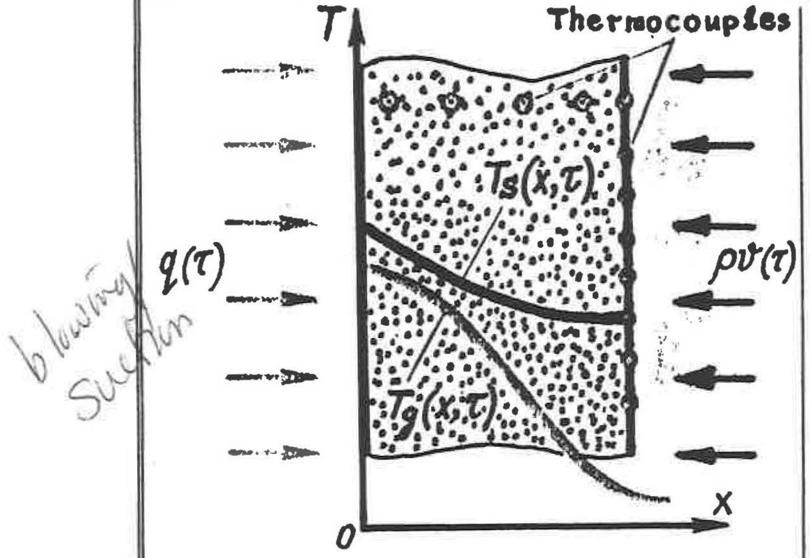
B_ν is a spectral radiation intensity of the absolutely black body;

θ_ν is a indicatrisse of dispersion; ω is a solid angle

Input data: temperature measurements in the body

$C(T)$ and $k(T)$ are unknown coefficients

3. Inverse problem of conductive - convective heat transfer



A · porous cooling :

1 are temperature - sensitive elements

Energy equations for solid (s) and gaseous (g) phases:

$$C_s \frac{\partial T_s}{\partial \tau} = \frac{\partial}{\partial x}\left(\boxed{k_s}\frac{\partial T_s}{\partial x}\right) - \frac{\boxed{h_v}}{1-P}(T_s - T_g),$$

porosity

$$(\rho C_p)\frac{\partial T_g}{\partial \tau} = \frac{\partial}{\partial x}\left(k_g\frac{\partial T_g}{\partial x}\right) - (\rho v C_p)_g\frac{\partial T_g}{\partial x} + \frac{\boxed{h_v}}{P}(T_s - T_g),$$

$$x \in (0,b), \quad \tau \in (0,\tau_m];$$

☐ may be unknowns

Initial conditions:

$$T_s(x,0) = \xi_s(x), \quad T_g(x,0) = \xi_g(x);$$

Boundary conditions:

$$-\boxed{k_s}\frac{\partial T_s(b,\tau)}{\partial x} = \boxed{h_0}[T_s(b,\tau) - T_{g0}], \quad -k_s\frac{\partial T_s(0,\tau)}{\partial x} = \boxed{q(\tau);}$$

$$(\rho v C_p)_g T_g(b,\tau) = (\rho v C_p)_g T_{g0} + h_0[T_s(b,\tau) - T_{g0}], \quad \frac{\partial^2 T_g(b,\tau)}{\partial x^2} = 0;$$

Darcy modified law : $$-\frac{dP_g}{dx} = \alpha(\mu v)_g + \beta(\rho v)_g^2;$$

Equation of state for gas : $$\rho_g = \boxed{P}_g M_g / 8314 T_g$$

pressure

where

C_p is a specific heat capacity at constant pressure;

v is velocity; P is body porosity; p is pressure;

ρ is density; μ is viscosity; M is a molecular weight;

α, β are hydraulic coefficients;

T_{go} is an initial temperature of the insufflated gas

Measurement data:

$$T_s(d_i,\tau)=f_i(\tau),\quad \tau\in[0,\tau_m],\quad i=\overline{1,N},\quad N\geq 1,\quad 0\leq d_1<d_2<\ldots<d_N\leq \delta$$

Unknown causal characteristics:

-- heat fluxes $q(\tau)$,

-- thermal conductivity $k_s(T_s)$,

-- inner heat transfer coefficiients h_v of the porous body,

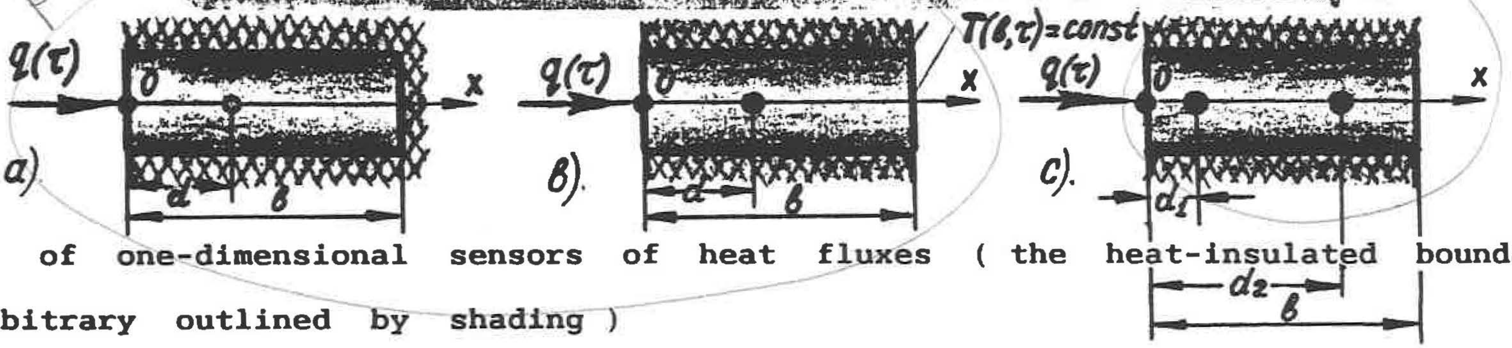
-- heat transfer coefficients h_0 at a coolant inlet into a porous body

IV.

Practical Applications of Inverse Heat Transfer Methods

* NON-STATIONARY HEAT DIAGNOSTICS

General principles of onedimensional heat measuring



one measurement point

two measurement points

Schemes of one-dimensional sensors of heat fluxes (the heat-insulated boundaries are arbitrary outlined by shading)

The determination of functions and parameters included in the boundary conditions (h, R, A, ε) can usually be reduced to a boundary inverse problem.

Example: heat transfer coefficient $h(\tau)$ - ?

$$h(\tau) = q_w(\tau) / [T^*(\tau) - T_w(\tau)],$$

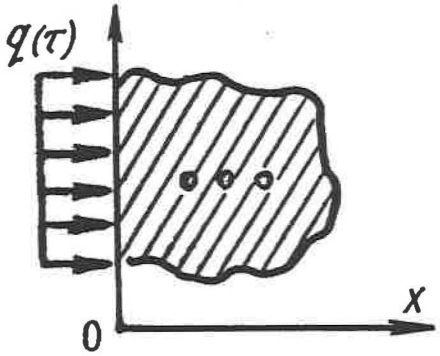
where T^* -- characteristic (recovery) temperature of gas (liquid);

T_w -- body surface temperature;

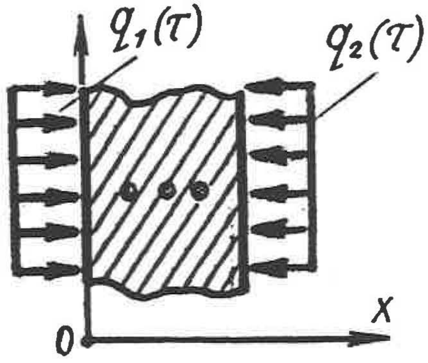
$$q_w(\tau) = q(\tau) + \varepsilon \sigma T_w^4(\tau)$$ ($q(\tau)$ and $T_w(\tau)$ are from I H C P)

TP
1

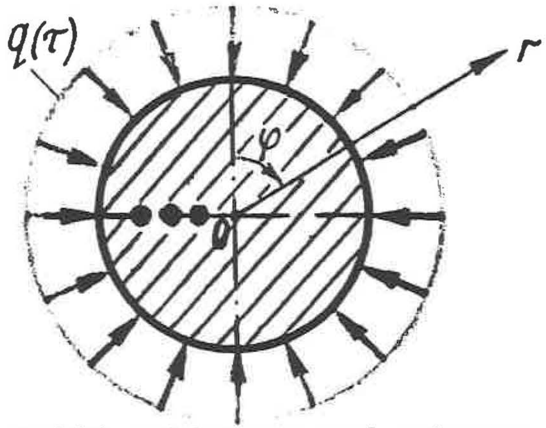
One-dimensional Statements of IHCP



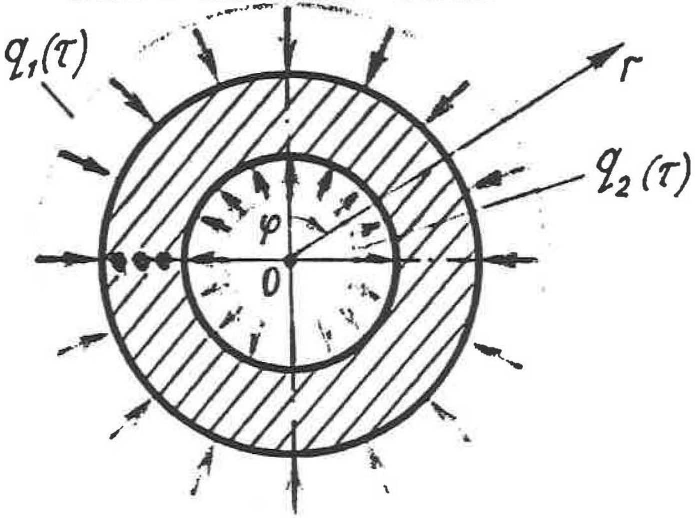
Semi - infinite Solid



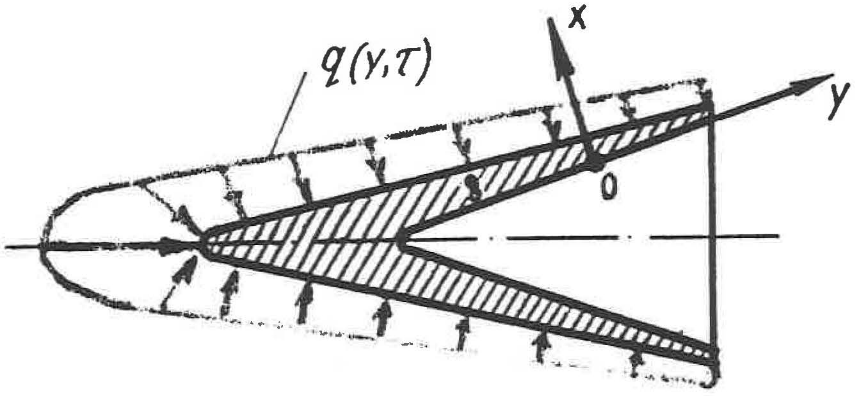
Slab



Solid cylinders and spheres



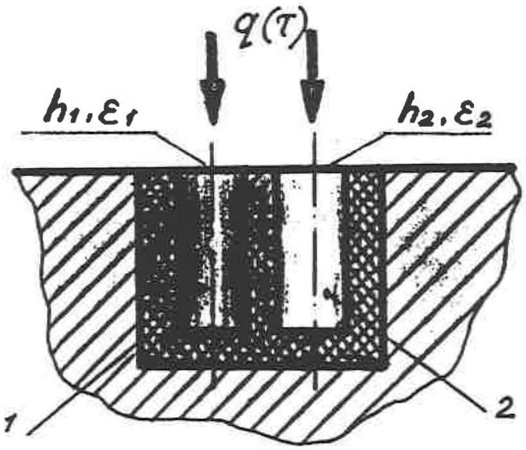
Hollow cylinders and spheres



Specimen with wall of variable thickness

○ ○ ○ - points of temperature measurements

Appl: Comparative testing technique



The scheme of a sensor with two sensitive elements 1 and 2

Unknown values:

- a local coefficient of heat transfer $h(T_W/T^*)$
- an integral coefficient of the surface radiation (emissivity) $\varepsilon(T_W)$

If $h_1(\tau) = h_2(\tau) = h(\tau), \quad \tau \in [0, \tau_m],$

common coating

and $\boxed{\varepsilon_1(\tau) = \varepsilon_2(\tau) = \varepsilon(\tau),} \quad \tau \in [0, \tau_m],$

a heat balance equations are

$$\text{(i)} \quad \begin{cases} h(T^* - T_{W_1}) = \varepsilon\sigma T_{W_1}^4 + q_1, \\ h(T^* - T_{W_2}) = \varepsilon\sigma T_{W_2}^4 + q_2, \quad \tau \in [0, \tau_m] \end{cases}$$

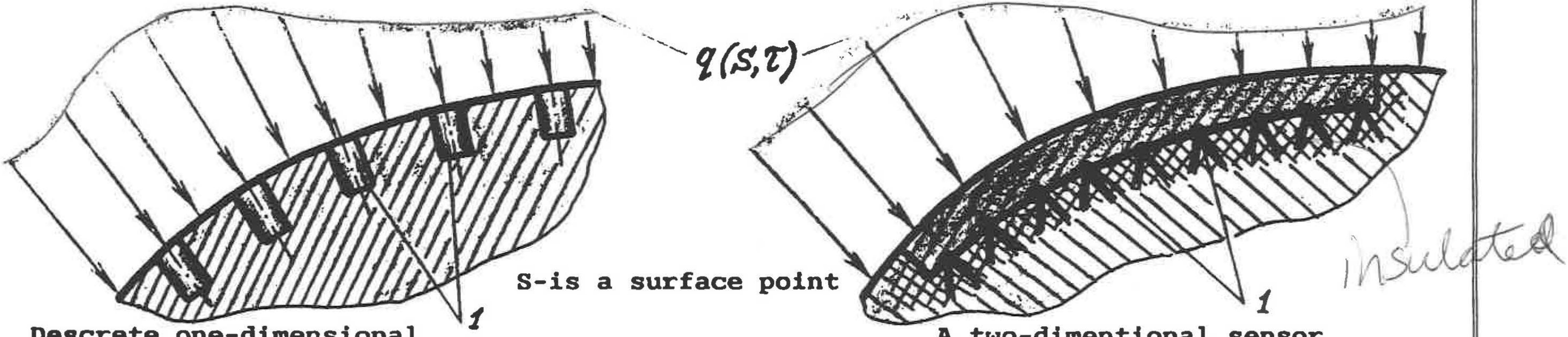
$T_{W_1}(\tau)$, $T_{W_2}(\tau)$, $q_1(\tau)$, $q_2(\tau)$ are obtained the solution of IHCP

Afterwards $h(\tau)$, $\varepsilon(\tau)$ are obtained from (i) for each moment τ

$$\{h(\tau), \varepsilon(\tau)\} \Rightarrow \{h(T_W/T^*), \varepsilon(T_W)\}$$

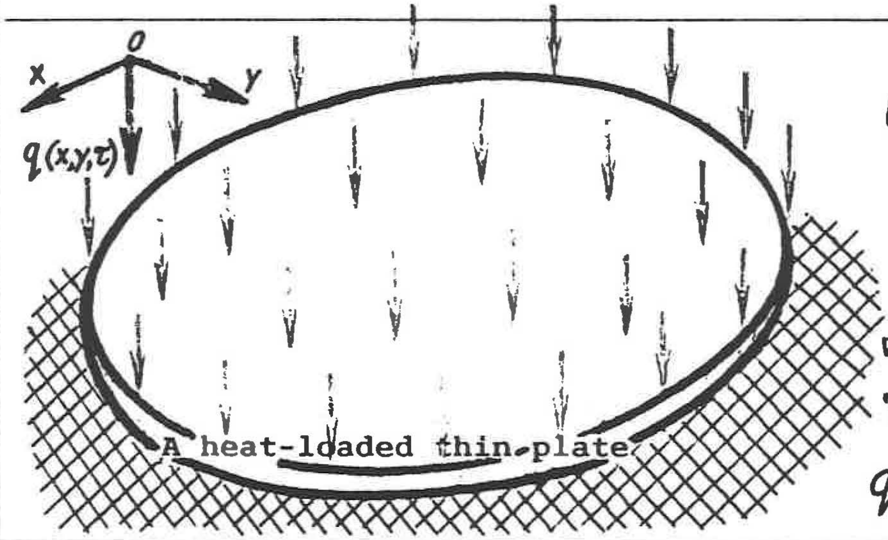
Add a third sensor, can introduce T^* as add'l unknown.

Appl: The diagnostics of spatial-time boundary conditions of heat transfer



Descrete one-dimensional
sensors (1) for reconstruction
of the spatial-time distribution
of heat fluxes

A two-dimentional sensor
of heat fluxes:
1 - thermocouples



$$c\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{q(x,y,\tau)}{b}$$

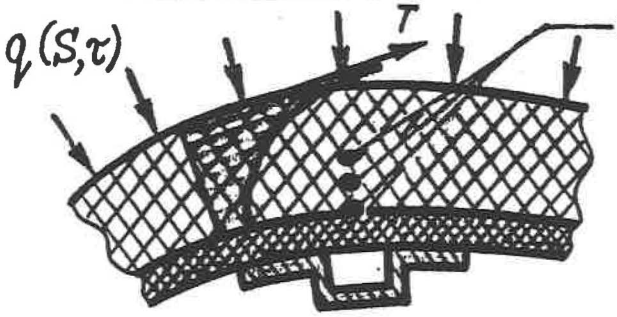
a source

Where b is a thickness of plate,

$T(x,y,\tau)$ is the measured temperature field,

$q(x,y,\tau)$ is an unknown value

THE RECONSTRUCTION OF TEMPERATURE FIELDS AND HEAT FLOWS

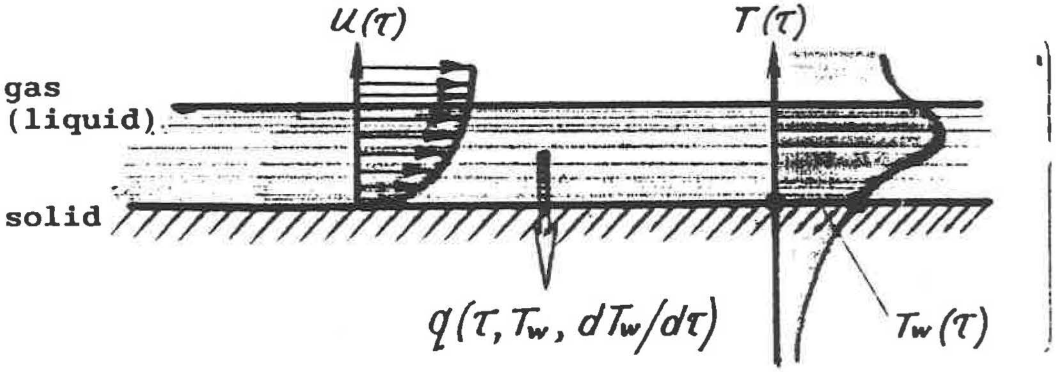


points of temperature measurements: $\{T_i(\tau)\}_1^N$

The study of thermo-strength of materials and structures

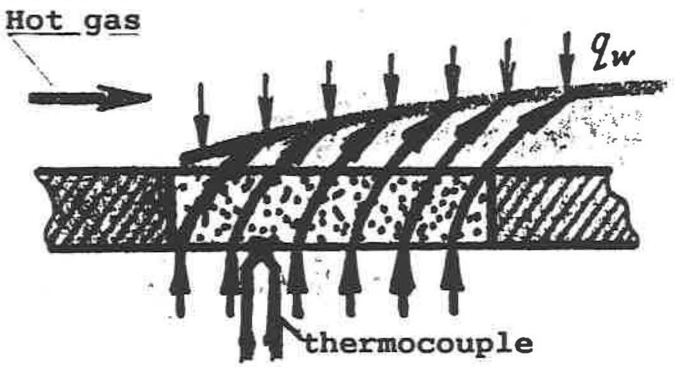
Example of a structure

THE STUDY OF NON-STATIONARY HEAT TRANSFER

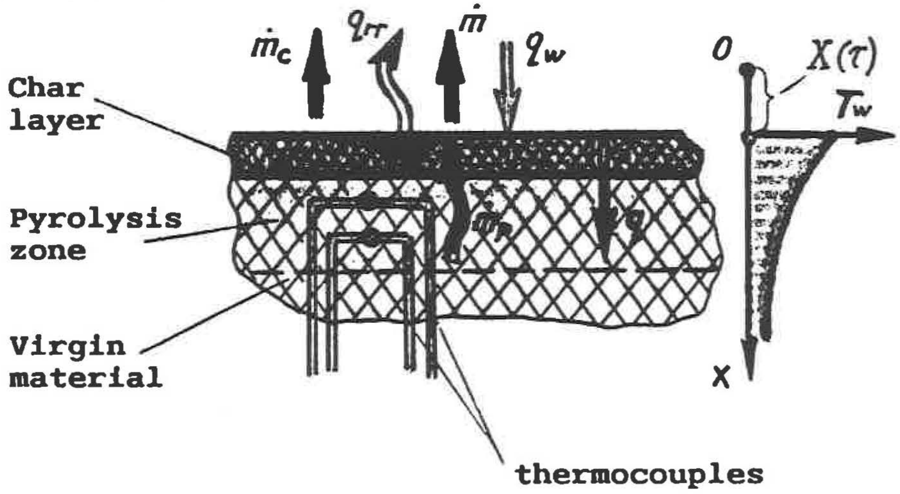


joint problem of heat-and mass transfer

Appl: The study of porous cooling, heating and destruction of materials



Porous cooling



Heat protection (char-forming) material

$$q_w^0 = q_{rr} + q + q_{ph.ch} + q_{Bl}$$

where q_w is a rate of heat transfer to surface by convection and radiation (0 - with no account for injection),

$q_{rr} = \varepsilon \sigma T_w^4$ are radiative heat losses on the surface,

q is a net heat flux in solid at surface,

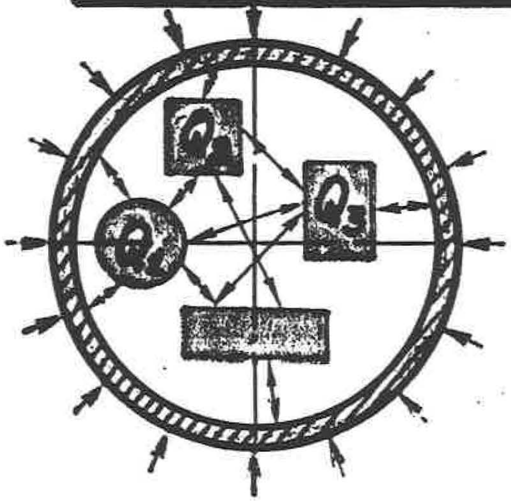
$q_{ph.ch}$ is a heat release or absorption due to physio-chemical processes,

q_{Bl} is a reduction of the surface heat transfer due to blowing,

$\dot{m}_c$ is a surface rate of ablation;

$\dot{m}_p$ is a rate of depolymerization of the virgin material

* IDENTIFICATION OF HEAT TRANSFER PROCESSES IN ENGINEERING DEVICES



Identification and correction of mathematical models of engineering systems:

* integral factors of absorption A and hemispherical emittance ε of the outer surface;

* coefficients for conduction and / or convection between the selected elements of the module;
  and so on

* DETERMINATION OF THERMAL PROPERTIES OF MATERIALS

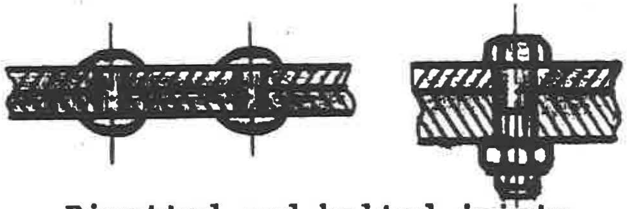
* determination of thermophysical properties and kinetic characteristics of heat-shield materials;

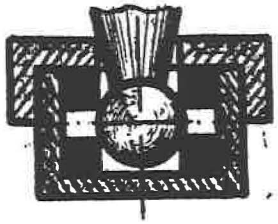
* determination of temperature dependence of heat conduction coefficient of a cooling ingot during steel tempering;

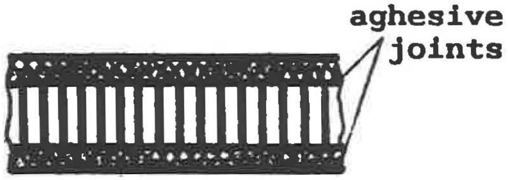
* determination of properties of freezing-and-melting soils;

and so on

* CONTACT HEAT TRANSFER STUDIES


Rivetted and bolted joints


Hinged joints



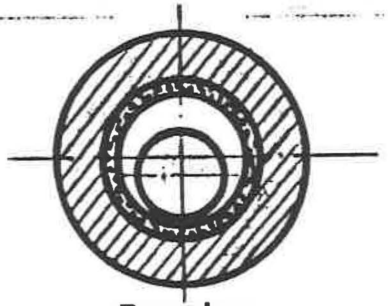
Multi-layer
thermal shilds

* EVALUATION AND MONITORING OF THE OPTIC RADIATION CHARACTERISTIES

Optic radiation properties of the spacecraft outer surface, radiators, solar batteries ets.

on end
of
Short Course

* DETERMINATION OF HEAD LOADINGS DURING FRICTION

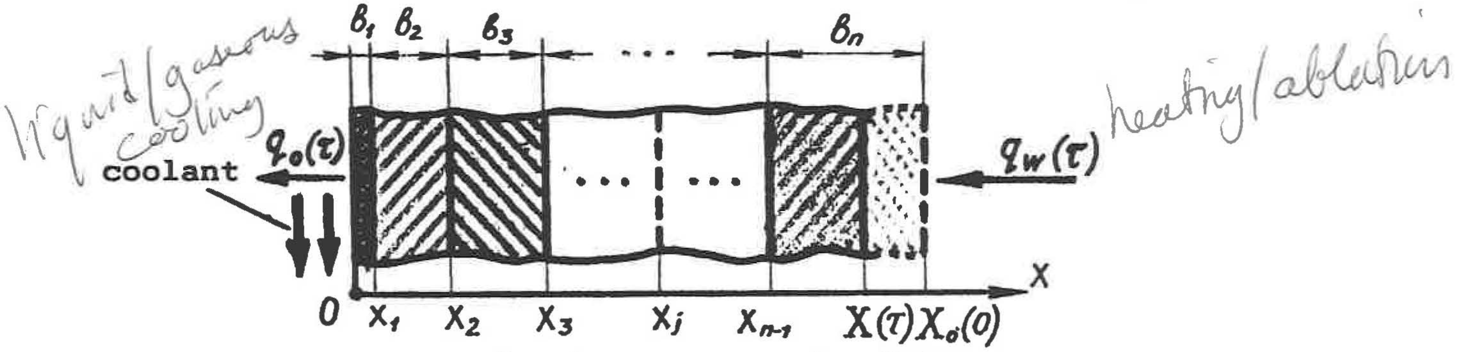

Bearing

* heat release during friction of two solid surfaces;

* heat trasfer influence in the friction zone on friction mechanism;

* friction moment

One-dimensional thermal mathematical model for n-layers H S



Scheme of HS

$$C_j(T)\frac{\partial T_j}{\partial \tau} = \frac{\partial}{\partial x}\left(k_j(T)\frac{\partial T}{\partial x}\right), \qquad (x,\tau)\in D_j ,$$

$$D_j = \{(x,\tau) : x_{j-1} < x < x_j ;\ x_0 = 0 ;\ 0 < \tau \le \tau_m\},\ j = \overline{1, n-1} ;$$

generalized heat conduction equation

$$C_n(T,\gamma)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(k_n(T,\gamma)\frac{\partial T}{\partial x}\right) + P(T,\gamma)\frac{\partial T}{\partial x} + S(T,\gamma), \quad (x,\tau)\in D_n ,$$

$$D_n = \{(x,\tau) : x_{n-1} < x < X(\tau) ;\ 0 < \tau \le \tau_m\},$$

where γ is a parameter taking into account dynamic character of changes in the properties of material during heating and destructing (for example, $\gamma = \frac{\partial T}{\partial \tau}$);

$X(\tau)$ is a moving boundary of a shield

Design of H S (continuation 1)

Known

Initial conditions : $T_j(x,0) = \xi_j(x), \quad j = \overline{1,n};$

Boundary conditions :

left end

$$k_1 \frac{\partial T_1}{\partial x}\bigg|_{x=0} + q_0(\tau) = 0, \qquad k_n \frac{\partial T_n}{\partial x}\bigg|_{x=X(\tau)} + q_n(\tau) = 0,$$

right end

$$q_0(\tau) = h\,[T_1(0,\tau) - T_{coolant}\,],$$

$$q_n(\tau) = q_w(T_w(\tau),\tau) - \varepsilon\sigma T_w^4(\tau) - q_{ph.ch}(T_w(\tau),\tau),$$

where h is a heat transfer coefficient; $T_w(\tau) = T_n(X(\tau),\tau);$

$q_{ph.ch}$ is a heat release or absorbtion due to phisico-chemical processes on surface

possible temp jump across layers (due to resistance)

Conjuction conditions :

$$T_j(x_j-0,\tau) = T_{j+1}(x_j+0,\tau) - R_j k_j \frac{\partial T_j}{\partial x}\bigg|_{x=x_j-0}, \quad j = \overline{1,n-1}$$

$$k_j \frac{\partial T_j}{\partial x}\bigg|_{x=x_j-0} - k_{j+1}\frac{\partial T_{j+1}}{\partial x}\bigg|_{x=x_j+0} = 0, \quad j = \overline{1,n-1}$$

Design of H S (continuation 2)

Unknown solution :

{ number of layers; types of materials; thinknesses of layers $\delta_2, \ldots, \delta_n$ }

Criterion : Specific mass of HS at given point τ of a surface

$$\boxed{m_\tau = \sum_{j=2}^{n} \rho_j \delta_j^\tau},$$

where ρ_j is a material density of layer j ;

δ_j^τ is thickness of a layer j at a point τ

Restrictions :

- on admissible temperatures of layers $T_j(x,\tau) \leqslant T_j^{max}$;
- on ranges of change of unknown parameters;
- on coolant heat capacity $\int_0^{\tau_m} q_0 \, d\tau \leqslant Q^{max}$

This problem is a combined coefficient - geometric inverse problem

TWO SUBPROBLEMS:

1. Determination of the thicknesses vector $\vec{B}_r = \{ B_j^r \}_{j=2}^{n}$ of layers for given materials and number of layers (geometric inverse problem)

$$\vec{B}_r : \min_{\vec{B}_r \in G} m_r(\vec{B}_r),$$

$$G = \{ \vec{B} : g_i(\vec{B}, T(x,\tau,\vec{B})) \leq 0 ; \ i = \overline{1,\ell} \},$$

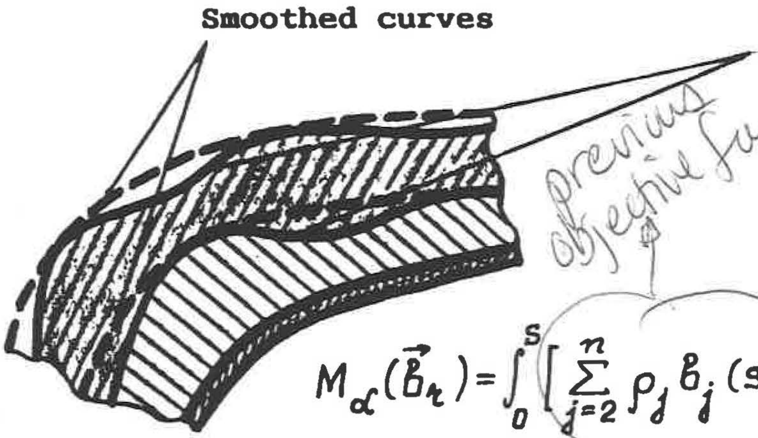
$$T(x,\tau,\vec{B}) = A[T(x,\tau,\vec{B}), \vec{B}] ; \ \vec{B} \in R^{n-1}$$

This subproblem may be solved by penalty functions method in combination with gradient methods

take to 1,

2. Determination of suitable materials and number of layers (this subproblem may be solved by the trial-and-error method)

Design of H S (continuation 4: Regularization of geometric inverse problem)



Smoothed curves

Values of thicknesses, obtained by solving above-stated problem

Modified criterion:

$$M_{\alpha}(\vec{B}_r)=\overbrace{\int_0^S\Big[\sum_{j=2}^{n}\rho_j B_j(s)\Big]ds}^{\text{previous objective function}}+\overbrace{\sum_{j=2}^{n}\alpha_j\int_0^S\Big[x_j^{(\gamma_j)}(s)\Big]^2 ds,}^{\text{smoothing term (regularizer)}}\quad \alpha_j>0,\ j=\overline{2,n}$$

where S is a contour length of support (i.e., at x = 0) surface;

γ_j is a degree of derivative choosed for j-th layer

Simplification : $\alpha_2=\alpha_3=\dots\alpha_n\equiv\alpha;\quad \gamma_2=\gamma_3=\dots=\gamma_n=1$ or 2

Choice of parameter α : $\alpha_{opt}:\ \big|M(\alpha)-M_{min}\big|-\Delta M_a\rightarrow\min_{\alpha}$, ← trial & error?

where $M(\alpha)=\int_0^S\Big[\sum_{j=2}^{n}\rho_i B_j^{\alpha}(s)\Big]ds;$

ΔM_a - maximum admissible deviation of linear mass of H S from the linear mass, corresponding to initial criterion m_r

V.

General Operator Form of an Inverse Problem and a Little on the History of the Subject Matter

Difficulties, arising when solving inverse problems

General formulation of an inverse problem in the form of operation equation is

$$\boxed{Au = f, \qquad u \in U, \qquad f \in F}, \qquad (i)$$

where u and f are unknown and observed elements respectively (vectors , function or vector - functions) ,

U , F are metric spaces ,

A - is a continuous operator .

The Hadamard conditions of well-posedness (1902) :

- Solution of problem (i) exists for any $f \in F$;
- it is unique in U ;
- it depends continuously on f .

existence
uniqueness
Stability

If at least one of the requirements is violated , the (i) problem is called ill-posed . This is the very situation , which is observed in solving the IHTP .

A Little on the History of the subject - matter

Problem	Reseachers	Time
* Historical climate and heat conduction of Earth 's ground layer	Fourier , Poisson , Kelvin	19th centuary
* Detemination of unsteady heat fluxes	Mirsepassi T.J.	1958
	Stolz G	1960
	Beck J.V.	1962 and later
	Aldoshin G.T. Golosov A.S. Zhuk V.I.	1968 and later
	Alifanov O.M.	1969 and later

first name is USA in inverse methods (by Oleg's accaunt)

Bousse-esq 1901

(CONTINUATION)

* Application of the principle of regular regime for heat fluxes recovery	Shumakov N.V.	1955 and later
* Pseudo - inverse heat conduction problem	Giedt W.H.	1955
	Kastelin O.N. Bronsky L.N.	1956
* Heat conduction problem in the Cauchy generalized formulation	Stefan J. – first exact solution to 1-D IHCP w/ constant coefficients	1890
	Tyomkin A.G.	1961
	Burggraf O.R.	1964

TP
1

On mathematical methods to a solution of ill-posed inverse problems

Method (Statement)	Researchers (Hadamard 1902)	Time
* Conditionally - well - posed problems	Carleman T. — first attempt to solve inverse problem	1926
	Tikhonov A.N. → formulated "conditionally well-posed" problem	1943
	Lavrentiev M.M.	1953 and later
	Ivanov V.K.	1956 and later
	John F. → Solves heat conduction ~~eq~~ equation w/ inverse time	1955 and later
* Integral equations of the first kind	Phillips D.L.	1962
	Twomey S.	1963 and later
* Regularization method (variational principle)	Felix, 1962 also **Tikhonov A.N.**	1963 and later
	Lavrentiev M.M., Ivanov V.K., Arsenin V.Ya., Morozov V.A., Bakushinsky A.B., Glasko V.B.	
* Iterative regularization	Alifanov O.M. Rumyantsev S.V. → mathematically grounded	1974 and later

universal - most effective with gradient method

monograph — ? Liones? Lyons

- for solving inverse heat conduction problems	Tikhonov A.N., Glasko V.B.	1967
	Alifanov O.M. Artyukhin E.A.	1971 and later
* Iterative reqularization method	Alifanov O.M. Artyukhin E.A. Rumyantsev S.V. Mikhailov V.V. Yudin V.M.	1974 and later
* Quasi - inversion method	Latt'es R., Lions J. - L.	1967
* Artifical hyperbolization method	Alifanov O.M.	1971

Heuristic reqularization

Approach	Researchers	Time
* Direct methods	**Shumakov N.V.,**	**1955 and later**
	Stolz G.	**1960**
	Beck J.V.	**1962 and later**
* Trial - and - error method	**Kozdoba L.A.**	**1968 and later**
* Linear dynamic filtration	**Symbirsky D.F.**	**1976 and later**
	Matsevitiy Yu. M.,	**1977 and later**
	Myltanovsky A.V.	

VI.

Principles of Solution for Ill-posed Problems

Principles of Solution for Ill-posed Problems

1. Conditionally well-posed statements
2. Approximation of the inverse operator A^{-1} by a family of continuous operators

---

$$Au = f$$

1. A conditionally well-posed problem statement:
   1) a solution $u \in M \subseteq D_A$ – the domain of the operator definition
   2) a solution is unique
   3) a solution is stable for $\tilde{f} = f + \Delta f \in M$

A class of well-posedness (acceptable solutions)

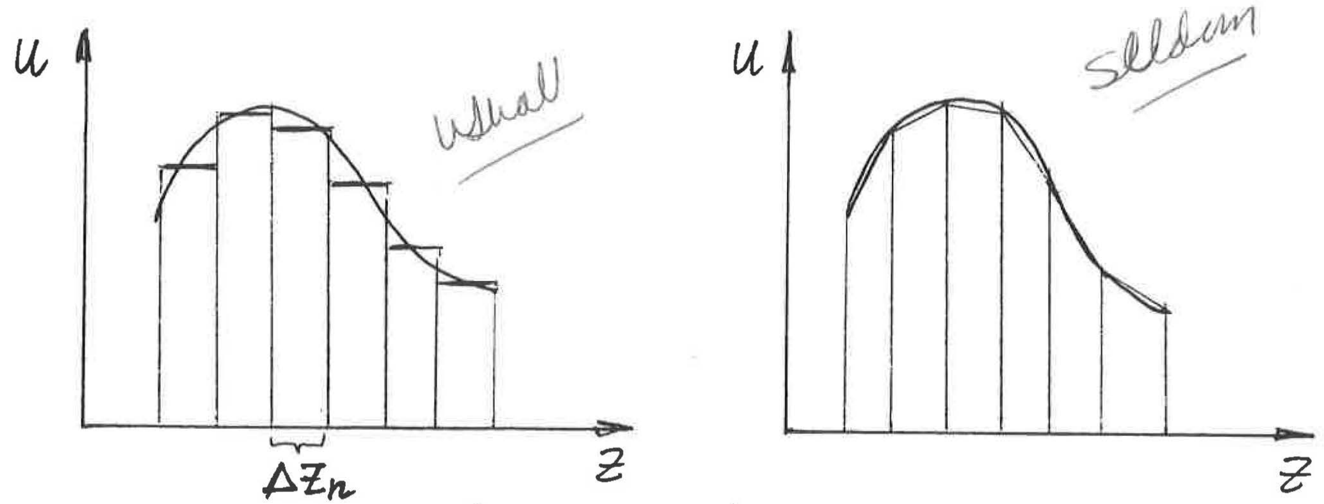
---

2. – self-regularization → by adjusting parameters in the solution (direct methods)
   – Tikhonov's regularization

Sources of the self (natural)-regularization:

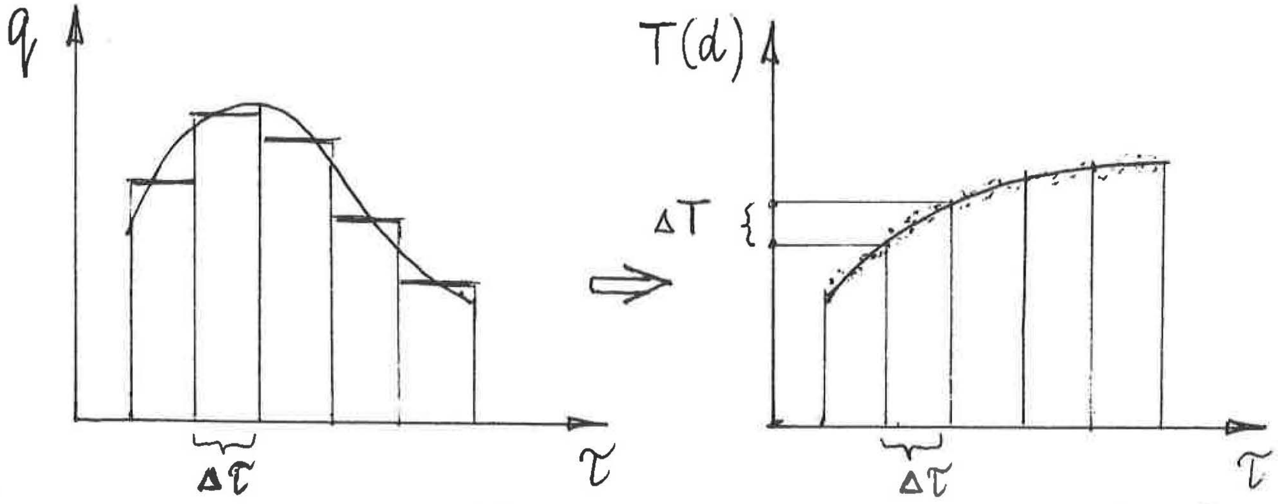
1) a regularization effect of heating
2) "viscous" properties of computational algorithms → natural filtration of high frequency components of solution

Step Regularization



Approximations

$$\Delta z_n = z_n - z_{n-1}, \quad n = 1, 2, \dots, N$$



$\Delta\tau$ must be chosen so that ΔT resulting will be significant relative to noise in the data

Viscous Properties of Algorithms

- a choice of approximation steps
- a choice of the degree of a polynomial, etc.

Want to compare residual error with expected error in data

VII.

Regularization of Unstable Problems by Tikhonov Method

Tikhonov Regularization Method

metric spaces

$$Au = f, \qquad u \in U, \; f \in F$$

A^{-1} is the inverse operator

If A^{-1} is bounded : $\quad u = A^{-1} f$

But A^{-1} is not obligatory continuous (!)

Domain of operator

Regularizing operator R_α : α is a scalar

1. $D_{R_\alpha} = F$, $\alpha > 0$; 2. R_α is continuous in F;

3. $R_\alpha A u \xrightarrow[\alpha \to 0]{} u$ $\quad \alpha =$ "regularizing parameter"

$\{A_h, f_\delta\}$ – are some approximations $\{A, f\}$

unknown

metric in space F

$$\rho_F(f_\delta, f) \leq \delta$$

(distance between members in space F)

$$d_U(A_h, A) \equiv \sup_{u \in U} \rho(A_h u, A u) \leq h (> 0)$$

d_u –

$\sigma = \{\delta, h\}$ is the error of input data

σ $\quad \delta > 0$

$h > 0$

$U_\alpha(\sigma) =$ an approximate solution

Regularizing operator:

1. $\exists\ \delta_0 > 0,\ h_0 > 0$: R_α is specified for $\alpha > 0$ and A_h, f_δ :

$$\rho_F(f_\delta, f) \le \delta \le \delta_0 , \quad d_{\dot{v}}(A_h, A) \le h \le h_0$$

2. $\exists\ \alpha = \alpha(\sigma)$: $\lim\limits_{\sigma \to 0} u_{\alpha(\sigma)} = \bar{u}$

Regularizing algorithm = Regularizing operator + Method for α selection

choice of α uses same information about the problem being solved.

Variational method

proposed by Tikhonov widely used in Russia

Smoothing functional

residual squared

stabilizing functio

$$\Phi_\alpha[u, A, f] = \rho_F^2(Au, f) + \alpha\Omega[u] , \quad u \in D(\Omega)$$

domain of function Ω.

where ρ_F is the residual in the space F;

$D(\Omega)$ is the domain of $\Omega[u]$;

$\Omega[u]$ is stabilizing functional:

$M_c = \{ u : \Omega[u] \le c \}$, $c \ge 0 =$ $=$ compact in U

compact with any sequence contains a subsequence.

Example: $\Omega = \rho_V(u, u^*)$,

$$\rho_U(u_1, u_2) \le \rho_V(u_1, u_2)$$

A is continuous, single-valued and additive;
$\Omega[u]$ is rigorously convex;
$\forall f \in F \;\exists\, u_\alpha \in D(\Omega): \min\limits_{u} \Phi_\alpha[u] = \Phi_\alpha[u_\alpha], \quad \alpha > 0$

Procedure for minimization of Φ_α
\+ a proper role for choosing α is
a regularizing algorithm for
an ill-posed problem

Iterative Regularization

is another effective method for solving ill-posed problems

Iteration number is the regularization parameter.

VIII.

Methods and Algorithms for Solving Inverse Heat Transfer Problems

Direct Semi-Analytical Methods

METHODS OF SOLVING ILL-POSED INVERSE PROBLEMS

Universal methods: a priori information of most general character

Examples: Tikhonov's regularization method;
Iterative regularization method

Problem-oriented methods: specific data on the problem

Example: Direct methods -

Boundary IHCP:

1. A boundary-value formulation.
2. The Cauchy formulation.
3. A variational formulation.

Direct Semi analytical
Direct Numerical methods
Regularization of semi-analytical forms
Regularization of numerical forms.
Iterative Regularization

Direct Semi-Analytical Method

Boundary IHCP

$$\frac{\partial T}{\partial \tau} = a \frac{\partial^2 T}{\partial x^2}, \quad x \in (0, b), \ \tau \in (0, \tau_m],$$

where $a = \text{const} > 0$.

$$T(x, 0) = \varphi(x), \quad x \in [0, b]$$

$$-\lambda \frac{\partial T(b, \tau)}{\partial x} = q^*(\tau), \quad \tau \in [0, \tau_m],$$

Known

where $\lambda = \text{const} > 0$.

$$T(d, \tau) = T^*(\tau), \quad \tau \in [0, \tau_m] \quad 0 < d \leqslant b$$

$$q(\tau) = -\lambda \partial T(0, \tau)/\partial x. \ - ?$$

object of investigation

Boundary-value statement of the inverse problem

Superposition principle and Duhamel's Integral

$$T(x, \tau) = \int_0^\tau q(\xi) \frac{\partial \vartheta(x, \tau - \xi)}{\partial \tau} d\xi + \int_0^\tau q^*(\xi) \frac{\partial \vartheta(b - x, \tau - \xi)}{\partial \tau} d\xi,$$

where

$$\vartheta(x, \tau) = \frac{1}{\lambda} \left\{ \frac{a\tau}{b} + \frac{3(b-x)^2 - b^2}{6b} + \frac{2b}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \times \right.$$

$$\left. \times \exp\left[-k^2 \pi^2 \frac{a\tau}{b^2}\right] \cos\left(k\pi \frac{b-x}{b}\right) \right\}.$$

Solution for step change in q''_{0v} $= 0$ with q_L

Integral form of IHCP:

$$Au \equiv \int_0^{\tau} u(\xi) K(\tau - \xi)\, d\xi = f(\tau), \quad \tau \in (0, \tau_m],$$

Volterra equation

Integral equation of first kind

↓

Solution well-known to be unstable

$$u(\tau) = q(\tau);\; K(\tau - \xi) = \partial \vartheta(d, \tau - \xi) / d\tau;$$

$$f(\tau) = T^*(\tau) - \int_0^{\tau} q^*(\xi)\, \frac{\partial \vartheta(b - d, \tau - \xi)}{\partial \tau}\, d\xi.$$

Instability of the problem:

$$u_\delta(\tau) = u(\tau) + \Delta u(\tau)$$

$$\Delta u(\tau) = C \sin \omega\tau$$

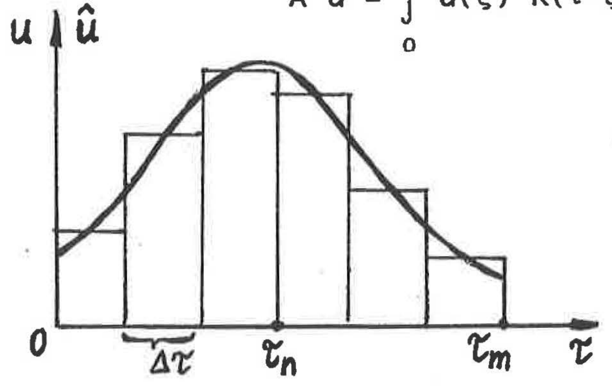
$$u(\tau) \in U, \quad u_\delta(\tau) \in U$$

$$\int_0^{\tau} u_\delta(\xi) K(\tau - \xi)\, d\xi \rightarrow \int_0^{\tau} u(\xi) K(\tau - \xi)\, d\xi$$

Approximation

from Mirsepassi Stoltz

$$A\,u \equiv \int_0^{\tau} u(\xi)\,K(\tau-\xi)\,d\xi = f(\tau), \quad \tau\in(0,\tau_m]$$



$$\Delta\tau_n = \tau_n - \tau_{n-1},\ n = \overline{1,m}$$

$$\hat{u}_n = u\left(\tau_n - \frac{\tau_n - \tau_{n-1}}{2}\right)$$

lower triangular matrix

$$\sum_{i=1}^{n} \hat{u}_i\,\Delta\vartheta_{in} = f_n, \quad n = \overline{1,m},$$

where $f_n = f(\tau_n)$, $\Delta\vartheta_{in} = \int_{\tau_{i-1}}^{\tau_i} K(\tau_n - \xi)\,d\xi$

$K(\tau-\xi)$ → time derivative of unit solution

$$T(x,\ \tau) = \int_0^{\tau} u(\xi)\,\frac{\partial\vartheta(x,\,\tau-\xi)}{\partial\tau}\,d\xi$$

$$\Delta\vartheta_{in} = \vartheta[d, \tau_n - \tau_{i-1}] - \vartheta[d, \tau_n - \tau_i]$$

result of ~~constant~~

Constant time steps: $\Delta\tau = \dfrac{\tau_m}{m}$

\# intervals in time domain

$$\Delta\vartheta_{1n} = \Delta\vartheta_{2,n+1} = \cdots = \Delta\vartheta_{m-n+1,m}$$

$$\sum_{i=1}^{n} \hat{u}_i \,\Delta\vartheta_{n-i} = f_n , \qquad n = 1, 2, \ldots, m$$

$$A_\Delta u = f$$

where

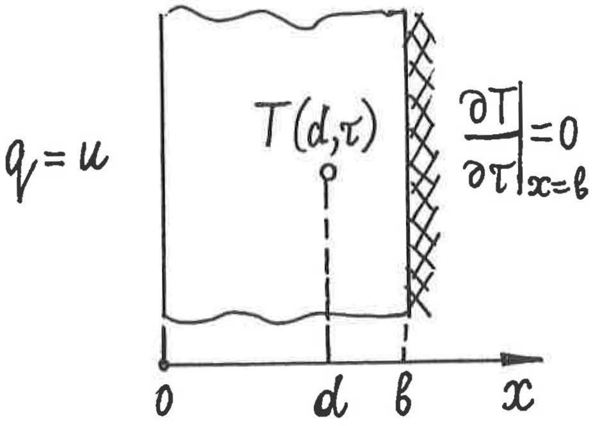
$$\Delta\vartheta_{n-i} \equiv \Delta\vartheta_{in} = \vartheta(d, \Delta\tau(n-i+1)) - \vartheta(d, \Delta\tau(n-i))$$

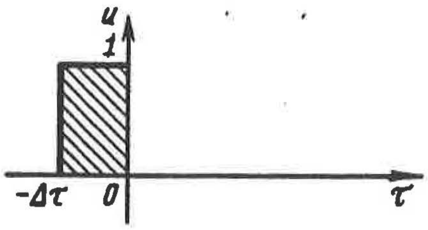
bi-diagonal equality

$$A_\Delta = \begin{bmatrix} \Delta\vartheta_0 & & & & & \\ \Delta\vartheta_1 & \Delta\vartheta_0 & & & & \\ \Delta\vartheta_2 & \Delta\vartheta_1 & \Delta\vartheta_0 & & & \\ \vdots & & \ddots & \ddots & \ddots & \\ \Delta\vartheta_m & \Delta\vartheta_{m-1} & \cdots & \Delta\vartheta_2 & \Delta\vartheta_1 & \Delta\vartheta_0 \end{bmatrix}$$

$$\boxed{m \text{ coefficients instead of } \frac{m(m+1)}{2} \;!}$$

$$\det A_\Delta = (\Delta\vartheta_0)^m$$





$$\Delta\vartheta_{n-i} = T(d, \Delta\tau(n-i))\Big|_{q=u,\ \frac{\partial T}{\partial\tau}\big|_{x=b}=0}$$

$$\det A_\Delta > 0 \Rightarrow A_\Delta^{-1}$$

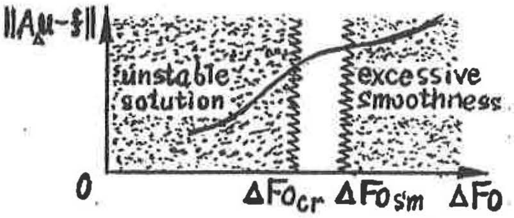
$$\boxed{\hat u_n = \frac{1}{\Delta\vartheta_0}\left(f_n - \sum_{i=1}^{n-1}\hat u_i \,\Delta\vartheta_{n-i}\right), \quad n=\overline{1,m}}$$

recurrant relation for upper triangular form

$$\Delta\vartheta_{n-i} \xrightarrow[\Delta\tau\to 0]{} 0 \quad (!)$$

Unstable for small $\underline{\Delta\tau}$

STEP REGULARIZATION PRINCIPLE



Critical step values (the kernels have maximums):

$$\Delta Fo_{\ell cr} = Fo_\ell^*,$$

thermal diffusivity

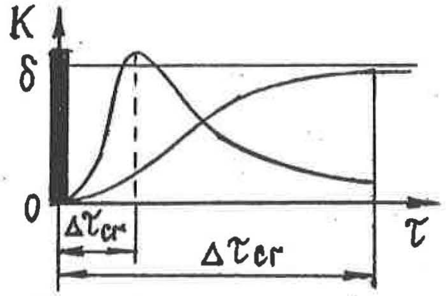
$$\boxed{Fo_\ell^* = \frac{a(\tau-\xi)^*}{\ell^2}},$$

characteristic length

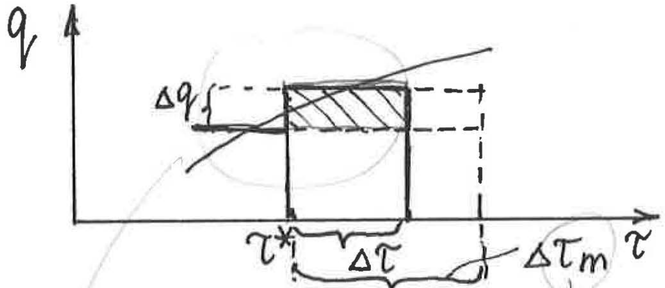
$\ell = d$ or b

plate = L
semi-infinite = d)

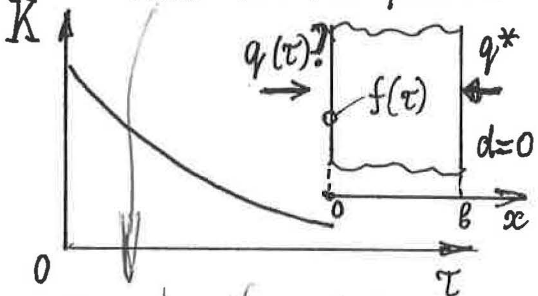
$$\Delta Fo_\ell > \Delta Fo_{\ell cr}$$



Kernels of the integral equation

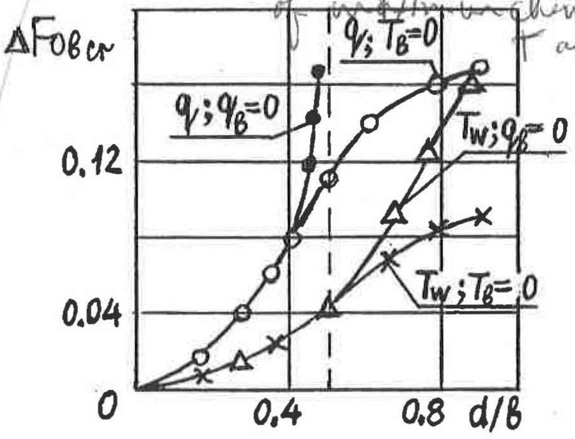


Pseudo-inverse problem



Construct q(t) at surface based on $T_0(t)$ measured

Corresponds to the point of maximum change in T at point d'.



ΔFo_{cr} for a plate with a thickness b

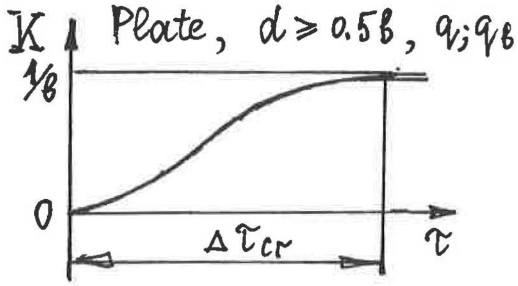
$$\Delta Fo_b > \Delta Fo_{b cr}$$

for $\Delta\tau < \Delta\tau_m$, data w/ noise, can get oscillatory behavior ~~even~~ with the "exact" data

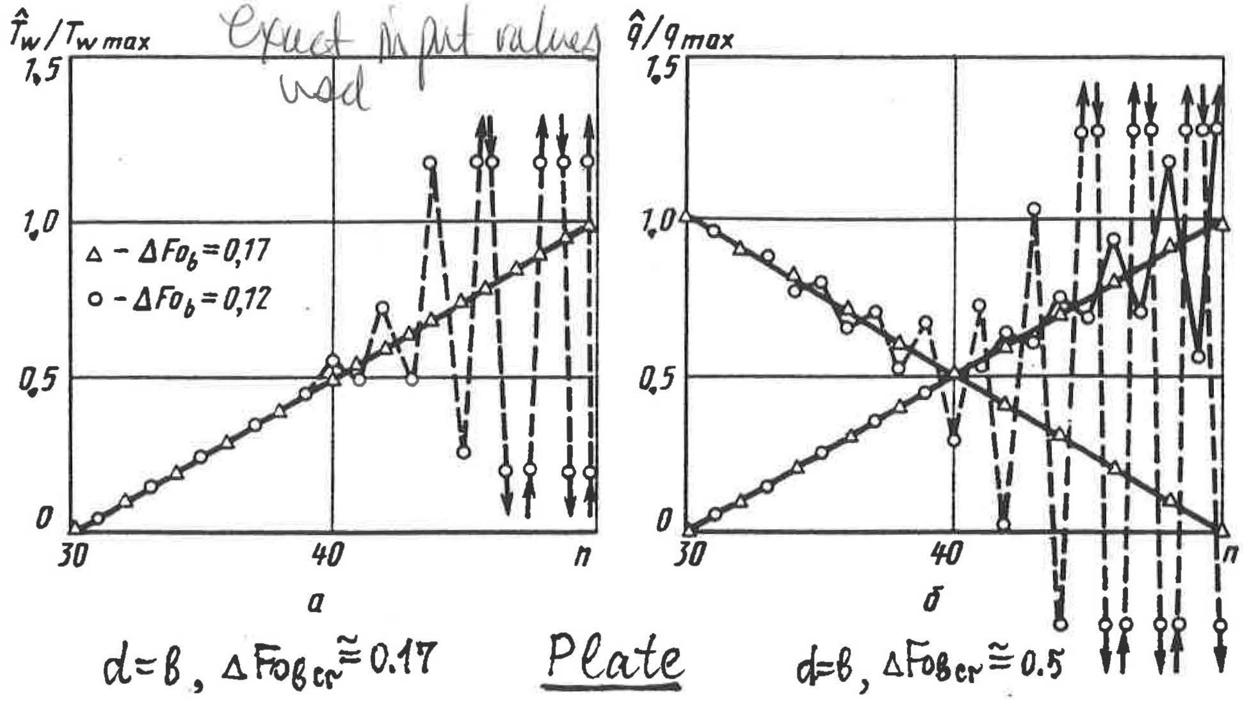
This method of finding $\Delta Fo_{B cr}$ only applies to T's with a maximum pure data, lit,

If a kernel does not reach it maximum at the finite time:

$$\boxed{\Delta \tau_{cr} = \ell^2 \, Fo^{**}/a}$$



Sensor at x=L.



Exact input values used

d=b, ΔFo_b cr ≅ 0.17

Plate

d=b, ΔFo_b cr ≅ 0.5

Perturbated Input Data

either

1. Preliminary smoothing of the input temperature data

OR

2. Matching of a temperature residual with the measurement error

$$m = \frac{\tau_m}{\Delta\tau} : \int_0^{\tau_m} \left[T(d,\tau,q_m) - f(\tau)\right]^2 d\tau \simeq \delta_f^2,$$

where $\delta_f^2 = \int_0^{\tau_m} \sigma^2(\tau)\,d\tau$

Smoothing of experimental data by cubic splines

then can interpolate to find T_i (spline)

Two-dimensional IHCP

General form of the recurrent algorithms:

square matrix

$$u_n = A_0^{-1}\left(f_n - \sum_{i=1}^{n-1} A_{n-i} u_i \right), \quad n = \overline{1, m}, \qquad (1)$$

intervals in time

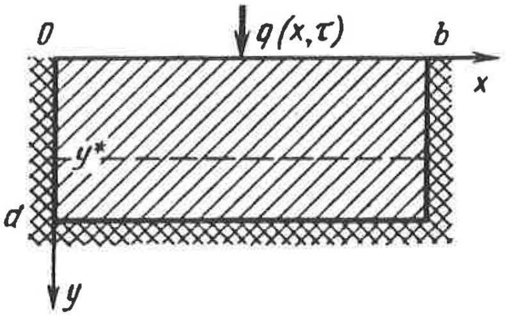
where u_n are vectors of unknown parameters;

f_n are known vectors;

A_s, $s = \overline{0, m-1}$ are square matrixes;

m is a number of time intervals.

Two-dimensional Plate



$$\frac{\partial T}{\partial \tau} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right), \quad x \in (0, b), \quad y \in (0, d), \quad \tau \in (0, \tau_m];$$

$$T(x, y, 0) = \psi(x, y), \quad x \in [0, b], \quad y \in [0, d];$$

$$-\lambda \frac{\partial T(x, 0, \tau)}{\partial y} = q(x, \tau), \quad \tau \in [0, \tau_m]$$ – unknown function;

$$\frac{\partial T(x, d, \tau)}{\partial y} = \frac{\partial T(0, y, \tau)}{\partial x} = \frac{\partial T(b, y, \tau)}{\partial x} = 0, \quad \tau \in [0, \tau_m],$$

$$T(x, y^*, \tau) = f(x, \tau), \quad \tau \in [0, \tau_m], \quad 0 < y^* \leqslant d$$

Integral form of the inverse problem:

$$\frac{a}{\lambda} \int_0^\tau d\xi \int_0^b q(x', \xi)\, G(x, y^*; x', 0; \tau - \xi)\, dx' = \bar{f}(x, y^*, \tau), \qquad (4.7)$$

Volterra equation of first kind

$$\tau \in (0, \tau_m], \quad x \in (0, b),$$

where

$$\bar{f}(x, y^*, \tau) = f(x, \tau) - \int_0^b dx' \int_0^d \psi(x', y')\, G(x, y^*; x', y'; \tau - \xi)\, dy';$$

green function

$$G(x, y; x', y'; \tau - \xi) = \frac{1}{bd} \left\{ 1 + 2 \sum_{r=1}^{\infty} \cos \frac{r\pi x}{b} \cos \frac{r\pi x'}{b} \times \right.$$

$$\left. \times \exp\left[-\frac{ar^2\pi^2(\tau - \xi)}{b^2}\right] \right\} \left\{ 1 + 2 \sum_{r=1}^{\infty} \cos \frac{r\pi y}{d} \cos \frac{r\pi y'}{d} \times \right.$$

$$\left. \times \exp\left[-\frac{ar^2\pi^2(\tau - \xi)}{d^2}\right] \right\},$$

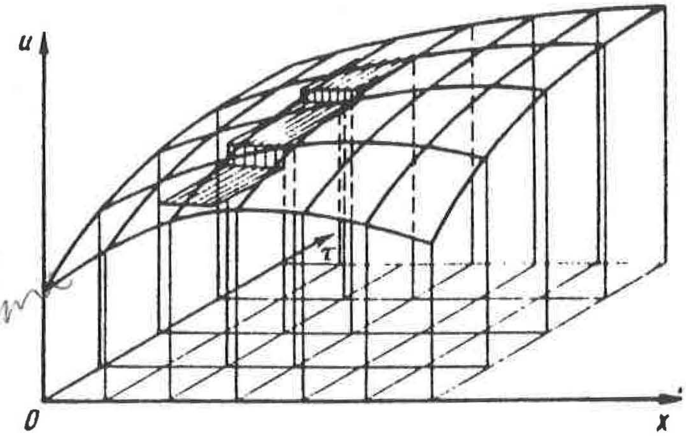
where

$G(x, y; x', y'; \tau - \xi)$ is the Green's function

Step Approximation

$x_l = l\Delta x,\ \ l = \overline{0, L};\ \ \tau_n = n\Delta\tau,$

$n = \overline{0, m}$



stepwise discrete space/time

$$\mathbf{u} = [\tilde{u}_{11}, \tilde{u}_{21}, \ldots, \tilde{u}_{L1};\ \tilde{u}_{12}, \tilde{u}_{22}, \ldots, \tilde{u}_{L2}; \ldots; \tilde{u}_{1m}, \tilde{u}_{2m}, \ldots,$$

$$\tilde{u}_{Lm}]^{\text{т}},\ \text{где}\ \ \tilde{u}_{ln} = \frac{\hat{u}_{l,n-1} + \hat{u}_{ln}}{2};\quad \hat{u}_{ln} = \frac{u_{l-1}(\tau_n) + u_l(\tau_n)}{2};$$

$$u_l(\tau_n) = u(x_l,\ \tau_n)$$

$$\mathbf{f}^* = [f^*_{11}, f^*_{21}, \ldots, f^*_{L1}; f^*_{12}, f^*_{22}, \ldots, f^*_{L2}; \ldots; f^*_{1m}, f^*_{2m}, \ldots, f^*_{Lm}]^{\text{т}}$$

$$\mathbf{A}_\Delta \mathbf{u} = \mathbf{f}^*$$

block matrices

$$\mathbf{A}_\Delta = \begin{bmatrix} \mathbf{A}_{11} & & & & \\ \mathbf{A}_{21} & \mathbf{A}_{22} & & & \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33} & & \\ \vdots & & & \ddots & \\ \mathbf{A}_{m1} & \mathbf{A}_{m2} & \cdots & \cdots & \mathbf{A}_{mm} \end{bmatrix}$$

$$\mathbf{A}_{11} = \mathbf{A}_{22} = \ldots = \mathbf{A}_{mm} \equiv \mathbf{A}_0;$$

$$\mathbf{A}_{21} = \mathbf{A}_{32} = \ldots = \mathbf{A}_{m,\,m-1} \equiv \mathbf{A}_1;$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mathbf{A}_{m-1,\,1} = \mathbf{A}_{m2} \equiv \mathbf{A}_{m-2}.$$

$$\mathbf{A}_S = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1L} \\ a_{21} & a_{22} & \ldots & a_{2L} \\ \vdots & \vdots & & \vdots \\ a_{L1} & a_{L2} & \ldots & a_{LL} \end{bmatrix}_S,$$

$$S = \overline{0, m-1}$$

bidiagonal equality

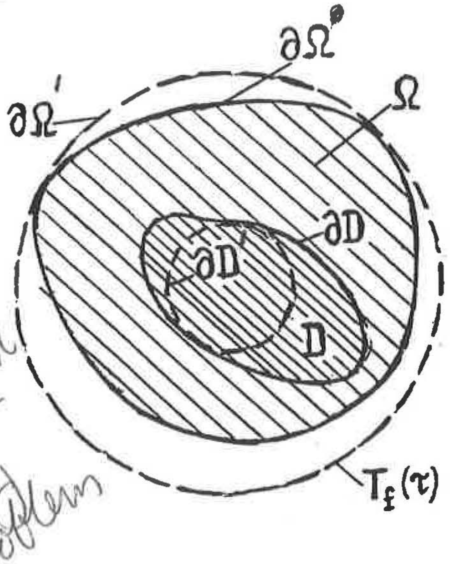
IX.

Methods for Solving Linear Inverse Heat Conduction Problems for Bodies with Movable Boundaries

METHOD OF FICTITIOUS BOUNDARIES

1974

Alifanov



Non characteristic Cauchy problem

Let domain Ω contains another domain D. Temperature $T(p, \tau)$ $p \in \partial D$ is known.

Four stages of the method:

1. Transfer from domain $\overline{\Omega}$ and $\overline{D}$ to new domains $\overline{\Omega'}$ and $\overline{D'}$:

$$\overline{\Omega} \subseteq \overline{\Omega'}, \quad \overline{D} \supseteq \overline{D'} \;;$$

$\overline{\Omega'} - \overline{D'}$ is the domain of simple form,

2. Solve an appropriate direct problem in $\overline{D'}$: $T(\tau)|_{\partial D'}, q(\tau)|_{\partial D'}$.

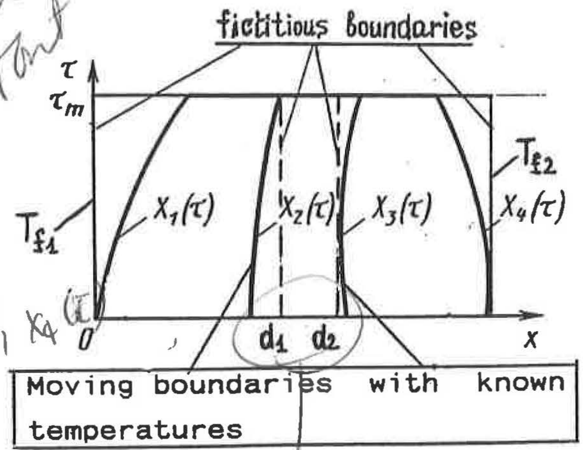
3. Solve of IHCP in $\overline{\Omega'} - \overline{D'}$: $T_f(\tau) = T(\tau)|_{\partial\Omega'}$.

4. Solve a direct problem in $\overline{\Omega'} - \overline{D'}$ and define the unknown boundary conditions at $\partial\Omega$.

Examples of application (bodies with movable boundaries)

thermal properties of body are constant

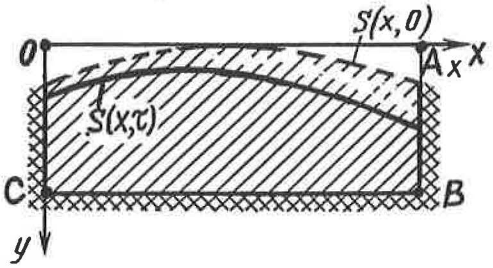
1. One-dimensional problem



fixed $X_1(\tau)$, $X_4(\tau)$

two fixed points

2. Two-dimensional problem



The rectangle OABC is domain $\overline{\Omega'}$

X.
Numerical Solution of Nonlinear IHCP

non linear IHCP

Direct Finite difference (step regularization)

SOLUTION OF BOUNDARY IHCPs BY DIRECT NUMERICAL METHODS

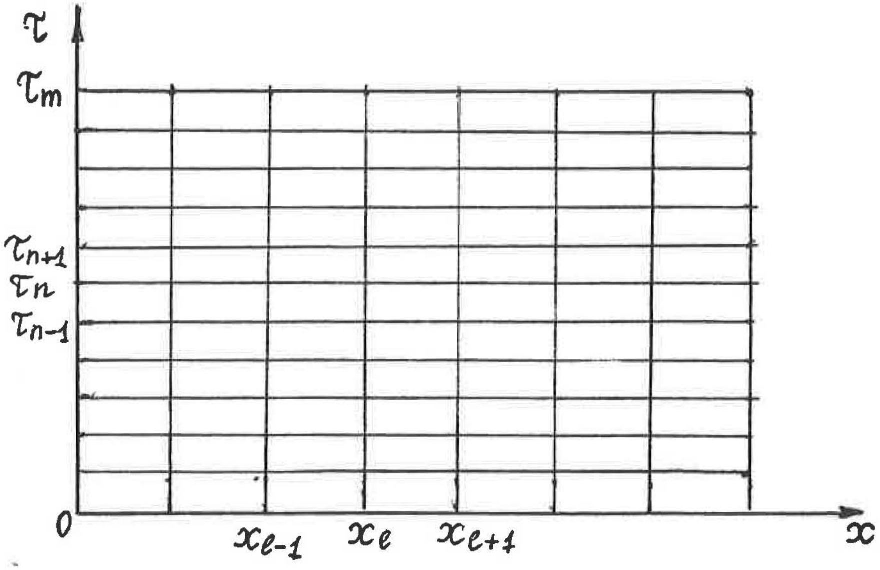
$$C(T)\frac{\partial T}{\partial \tau}=\frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right),\quad x\in(0,b),\quad \tau\in(0,\tau_m];$$

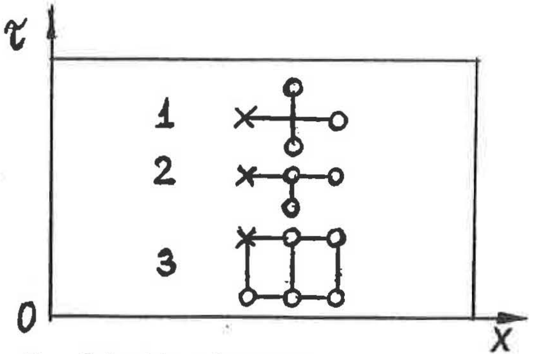
$$T(x,0)=\varphi(x),\quad x\in[0,b];$$

$$T(d,\tau)=f(\tau),\quad \tau\in[0,\tau_m];$$

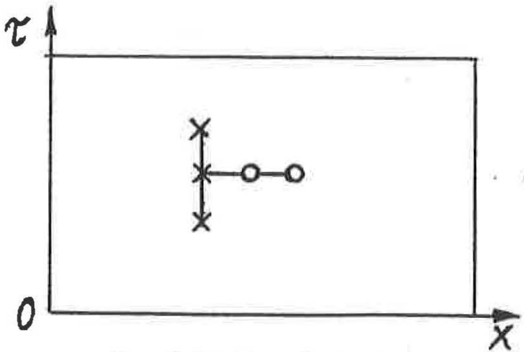
$$-\lambda(T(b,\tau))\frac{\partial T(b,\tau)}{\partial x}=q^*(\tau),\quad \tau\in[0,\tau_m],$$

where $C(T), \lambda(T), \varphi(x), f(\tau), q^*(\tau)$





Explicit shemes



Implicit sheme

O = Known

X = T.B.D.

Condition of <u>weak stability</u> of finite-difference solution of IHCP:

$$\| u_h \| \le C_h \| f_h \| , \qquad (5)$$

where C_h is a positive constant, depending on parameters of approximation.

Condition of the weak stability:

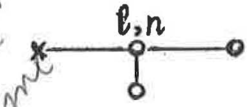
$\Delta \mathrm{Fo} \geq \Delta \mathrm{Fo}_{allowable}$

supplemental regularizing parameter

Scheme viscosity as a regularizing factor

The first differential approximation of a difference scheme:

"T-type" scheme



$$\frac{\partial T}{\partial \tau}\Big|_{ln} + \frac{\Delta\tau}{2}\frac{\partial^2 T}{\partial \tau^2}\Big|_{ln} = a\frac{\partial^2 T}{\partial x^2}\Big|_{ln}$$

Improves stability of inverse solution

non-characteristic Cauchy problem ?

Hyperbolic heat conduction equation:

$$\frac{w^2}{a}\frac{\partial T}{\partial \tau} + \frac{\partial^2 T}{\partial \tau^2} = w^2\frac{\partial^2 T}{\partial x^2}$$

Semi-infinite body, the first boundary-value problem

(Solution for hyperbolic equation)

$$T(1,\mathrm{Fo}) = T_W\left(\mathrm{Fo} - \sqrt{\frac{\Delta \mathrm{Fo}}{2}}\right)\exp\left[-\frac{1}{\sqrt{2\Delta \mathrm{Fo}}}\right] +$$

$$+\frac{1}{\sqrt{2\Delta \mathrm{Fo}}}\int_{\sqrt{\Delta \mathrm{Fo}/2}}^{\mathrm{Fo}} T_W(\xi)\exp\left[-\frac{\mathrm{Fo}-\xi}{\Delta \mathrm{Fo}}\right]\times$$

$$\times\frac{I_1\left(\dfrac{\sqrt{(\mathrm{Fo}-\xi)^2-\Delta \mathrm{Fo}/2}}{\Delta \mathrm{Fo}}\right)}{\sqrt{(\mathrm{Fo}-\xi)^2-\Delta \mathrm{Fo}/2}}\,d\xi,$$

Modified Bessel function of first kind

where $T_W(\mathrm{Fo})$ *is the surface temperature* $(T(0,\mathrm{Fo}) = T_W(\mathrm{Fo}))$;

$I_1(\cdot)$ *is the modified Bessel's function of the first kind*

$\boxed{\Delta \mathrm{Fo}_{cr} \simeq 0.04}$

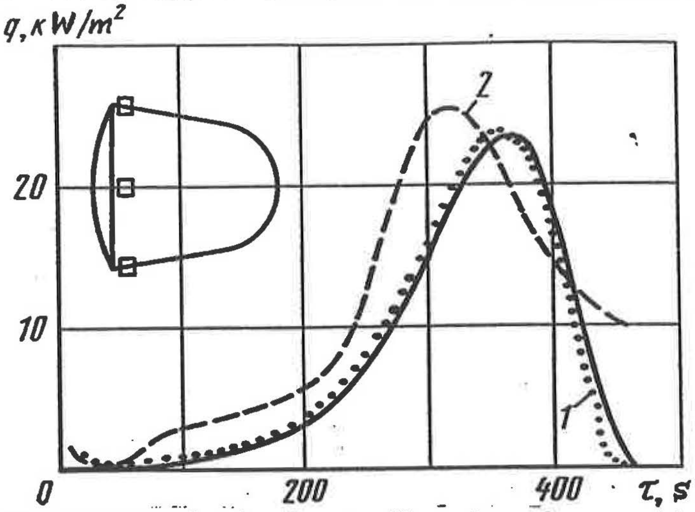
estimate of critical value for "T-type" discretization.

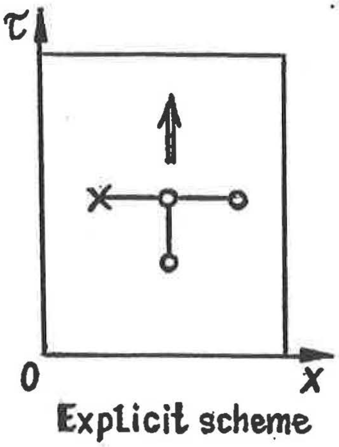
The implicit difference scheme possesses higher viscous regularizing properties then the explicit scheme:

$\boxed{\Delta \mathrm{Fo}_{cr} = 0.01 - 0.02}$

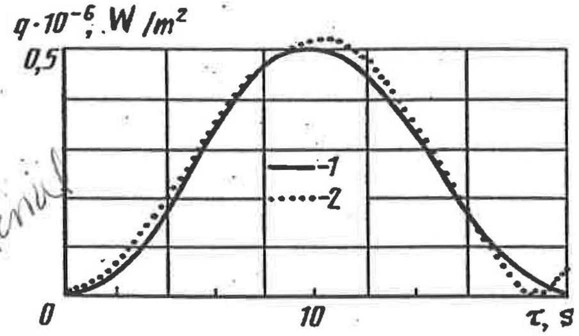
PRELIMINARY SMOOTHING OF THE INPUT TEMPERATURE DATA

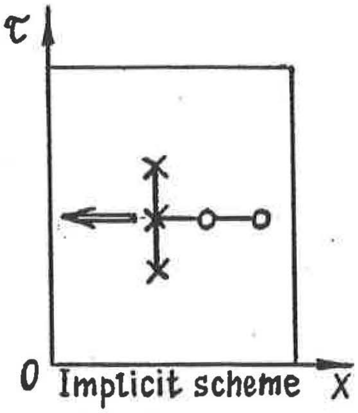
It is advisable to use algorithms of smoothing which give uniform approximation of the function and its derivatives.





Recovery of the heat flux by the explicit T-type scheme in mentioned points of re-entry vehicle: —— is a true solution. Error of input data $3\sigma(\tau) = 0.08 \cdot T_{max}$, $\Delta Fo_d = 0.05$; Results: ••••• is for smoothing $T_\delta(\tau)$ by the regularization method of the second order; — — — is for smoothing $T_\delta(\tau)$ by the least-squares technique (by "five points"), $N = 10^3$.





thermal properties used were for a graphite material

Recovery of the heat flux by the implicit scheme:

$b=0.002$; $a=0.4\ 10^{-6} - 0.143 \cdot 10^{-9}T + 0.408 \cdot 10^{-12}\ T^2, \frac{m^2}{s}$;

$\lambda = 0.721+0.288 \cdot 10^{-3}T, \frac{kw}{mK}$; —— is a true solution;

••••• is the results of IHCP solution, $\Delta\tau = 0.4$ s

XI.

Artificial Hyperbolization of Heat Conduction Equation

Artificial Hyperbolization of Heat Conduction Equation in Solving a Boundary Inverse Problem

noncharacteristic value cauchy problem

$$\frac{\partial T}{\partial \tau} + \alpha \frac{\partial^2 T}{\partial \tau^2} = a \frac{\partial^2 T}{\partial x^2}, \quad x \in (0, b), \quad \tau \in (0, \tau_m], \qquad (9)$$

proposed by Alifanov in 1971

where parameter α corresponds to fictitious relaxation time

Inverse problem: it is required to find function $T(x, \tau)$ if

$$T(x, 0) = 0, \quad 0 \le x \le b; \qquad \frac{\partial T(x, \tau_m)}{\partial \tau} = 0, \quad 0 \le x \le d;$$

$$T(d, \tau) = T^*(\tau), \quad -\lambda \frac{\partial T(b, \tau)}{\partial x} = q^*(\tau), \quad 0 \le \tau \le \tau_m,$$

hyperbolic is well posed where parabolic is ill posed

where $T^*(\tau)$ and $q^*(\tau)$ are known functions.

Condition for choice of α: $\left| \sum_{n=1}^{m} \left(T^{\alpha}_{x_n} - T^*_n \right)^2 - \delta^2 \right| = \min_{\alpha}$,

where $T^{\alpha}_{x_n}$ are temperatures, obtained from a solution of Fourier general heat conduction equation using a known boundary condition $q^*(\tau)$ and condition $q^{\alpha}(\tau)$ computed in in the result of IHCP solution for hyperbolic equation (9); δ is the known error of temperature data T^*.

XII.

Tikhonov Regularization Method

REGULARIZATION OF VARIATIONAL FORMS OF IHCP

Onedimensional problems

General integral form of linear boundary IHCPs:

$$\int_0^{\tau} u(\xi)K(\tau,\xi)d\xi = f(\tau), \quad \tau\in(0,\tau_m],$$

where u(τ) is the unknown function (heat flux, temperature of the surface, heat potential density).

$$Au=f, \quad u\in U, \quad f\in F$$

Hilbert spaces

A^{-1} can be unbounded

$$f=f_\delta, \quad \|\bar{f}-f_\delta\|_F \leqslant \delta$$

known error

$$\|Au-f_\delta\|_F \leqslant \delta \quad \Rightarrow \quad D_\delta$$

set of suspected solutions to

$\Delta = \|Au - f_\delta\|$ is a residual

$$u = u_\delta \in D_\delta \ : \quad \min_{u\in D_\delta} \|u-u^*\|_U$$

$U = W_2^K[0,\tau_m]$ is the Sobolev space

$f \in L_2[0,\tau_m]$

$$\boxed{\begin{gathered}\min_{u\in D_\delta}\|u-u^*\|^2_{W_2^k};\\ D_\delta=\{u:\|Au-f_\delta\|^2_{L_2}\leqslant\delta^2\},\end{gathered}}$$

continuous ad linear

Lagrangian Method

Lagrange multiplier

$$F[u,\ u^*,\ \lambda,\ \gamma]=\|u-u^*\|^2_{W^k_2}+\lambda(\|Au-f_\delta\|^2_{L_2}+\gamma^2-\delta^2_{L_2})$$

auxillary variable

$$\partial F[u,\ u^*,\ \lambda,\ \gamma]/\partial\lambda=0 \qquad \partial F[u,\ u^*,\ \lambda,\ \gamma]/\partial\gamma=0$$

$$\gamma=0$$

$$\lambda:\quad \|Au^\lambda-f_\delta\|^2_{L_2}=\delta^2_{L_2}$$

$$\alpha=\frac{1}{\lambda}\ :$$

$$\boxed{\begin{array}{c}\min\limits_{u\in W^k_2}\{\Phi[u,\ u^*,\ \alpha]=\|Au-f_\delta\|^2_{L_2}+\alpha\|u-u^*\|^2_{W^k_2}\}; \\ \|Au^\alpha-f_\delta\|^2_{L_2}=\delta^2_{L_2}. \qquad (A')\end{array}} \qquad (A)$$

residual known error

Condition (A') is <u>the principle (method) of residual</u>

Norm in L_2

$$\|z\|^2_{L_2}=\int_0^{\tau_m} z^2(\xi)d\xi:$$

Norm in W^k_2

$$\|z\|^2_{W^k_2}=\int_0^{\tau_m}\sum_{j=0}^{k}\left[\frac{d^jz(\xi)}{d\tau^j}\right]^2 r_j(\xi)\,d\xi,$$

$$\min_{u\in W^k_2}\{\Phi[u,\ u^*,\ \alpha]=\int_0^{\tau_m}d\tau\left[\int_0^{\tau}u(\xi)K(\tau,\ \xi)d\xi-f_\delta(\tau)\right]^2+ \qquad (B)$$

$$+\alpha\int_0^{\tau_m}\sum_{j=0}^{k}r_j(\xi)[u(\xi)-u^*(\xi)]^{(j)2}d\xi\Big\},$$

$$\alpha:\quad \int_0^{\tau_m}\left[\int_0^{\tau}u^\alpha(\xi)K(\tau,\ \xi)d\xi-f_\delta(\tau)\right]^2d\tau=\delta^2_{L_2} \qquad (C)$$

the number $k=1$ or 2 is usually taken.

Also, $r_j(\xi)=1$ $j=$ all

Euler equation for Φ :

$$\boxed{\begin{gathered}\int_0^{\xi} u(\zeta)K_1(\xi,\zeta)d\zeta+\int_{\xi}^{\tau_m} u(\zeta)K_2(\xi,\zeta)d\zeta-\bar{b}(\xi)+\\+\alpha\sum_{j=0}^{k}(-1)^j[r_j(\xi)(u(\xi))^{(j)}]^{(j)}=0,\end{gathered}}$$

where $K_1(\xi,\zeta)=\int_{\xi}^{\tau_m} K(\tau,\xi)K(\tau,\zeta)\,d\tau;$

$$K_2(\xi,\zeta)=\int_{\zeta}^{\tau_m} K(\tau,\xi)K(\tau,\zeta)\,d\tau;$$

$$\bar{b}(\xi)=\int_{\xi}^{\tau_m} f_{\delta}(\tau)K(\tau,\xi)\,d\tau.$$

Boundary conditions:

$$\sum_{j=p}^{k}(-1)^j[r_j(\xi)u^{(j)}(\xi)]^{(j-p)}\big|_{\xi=0;\tau_m}=0,\ p=1,\ 2,\ \ldots,\ k$$

$$\Downarrow$$

$$\{u^{\alpha}(\xi)\}\ \Rightarrow\ u_{\delta}^{\alpha}(\xi)$$

Can discretize and Solve → but another approach suggested by Alifanov →

Regularization of Semi-analytical Solution of IHCP

$$\sum_{i=1}^{n} \hat{u}_i \varphi_{ni} = f_{\delta n}, \; n = \overline{1, m}$$

$$\boxed{\mathbf{A}_\Delta \mathbf{u} = \mathbf{f}_\delta ,}$$

Same as before

where $\mathbf{A}_\Delta$ is the lower triangular matrix;

$$\mathbf{u} = \left\{ \hat{u}_i \right\}_1^m ; \qquad \mathbf{f}_\delta = \left\{ f_{\delta n} \right\}_1^m$$

If $k = 2, \; u^* = 0$; $r_0 = 0, r_1(\tau) = r_1 = \text{const} \geqslant 0, r_2(\tau) = r_2 = \text{const} \geqslant 0,$

$$\Phi_\alpha[u] = \sum_{n=1}^{m} \left(\sum_{i=1}^{n} \hat{u}_i \varphi_{ni} - f_{\delta n} \right)^2 + \frac{\alpha r_1}{\Delta\tau^2} \sum_{n} (\hat{u}_n - \hat{u}_{n-1})^2 +$$

$$+ \frac{\alpha r_2}{\Delta\tau^4} \sum_{n} (\hat{u}_{n+1} - 2\hat{u}_n + \hat{u}_{n-1})^2 .$$

$$\alpha : \quad \sum_{n=1}^{m} \left(\sum_{i=1}^{n} \hat{u}_i^\alpha \varphi_{ni} - f_{\delta n} \right)^2 = \delta^2_{\mathrm{E}m}$$

Choice determined by degree of smoothness and B.C's

$$(\mathbf{B} + \alpha \mathbf{C}) \mathbf{u} = \mathbf{d} + \mathbf{g}$$

where

$$\mathbf{B} : \quad b_{kl} = \sum_{n=l}^{m} \varphi_{nk} \varphi_{nl}, \quad b_{kl} = b_{lk}, \quad k, l = \overline{1, m}$$

$$\mathbf{d} : \quad d_k = \left\{ \sum_{n=k}^{m} f_{\delta n} \varphi_{nk} \right\}, \; k = \overline{1, m}$$

The first case:

a regularization of the first order

$$r_1/\Delta\tau^2 = 1, \quad r_2 = 0,$$

$$u_1 - u_0 = C_{10},$$

$$u_{m+1} - u_m = C_{1m};$$

$$C = \begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & & \ddots & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}$$

$$g = [-\alpha C_{10}, 0, \dots, 0, \alpha C_{1m}]^T;$$

The second case:

a regularization of the second order

$$r_1 = 0, \quad r_2/\Delta\tau^4 = 1,$$

$$u_0 - 2u_1 + u_2 = C_{20},$$

$$-u_{-1} + 3u_0 - 3u_1 + u_2 = C_{30},$$

$$u_{m-1} - 2u_m + u_{m+1} = C_{2m},$$

$$-u_{m-1} + 3u_m - 3u_{m+1} + u_{m+2} = C_{3m},$$

$$C = \begin{bmatrix} 1 & -2 & 1 & & & & & & \\ -2 & 5 & -4 & 1 & & & & & \\ 1 & -4 & 6 & -4 & 1 & & & & \\ & 1 & -4 & 6 & -4 & 1 & & & \\ & & & \ddots & & & & & \\ & & & & 1 & -4 & 6 & -4 & 1 \\ & & & & & 1 & -4 & 5 & -2 \\ & & & & & & 1 & -2 & 1 \end{bmatrix}$$

$$g = [\alpha(C_{20} + C_{30}), \; -2C_{20}, 0, \dots, 0, \alpha C_{2m}, \alpha(C_{3m} - C_{2m}$$

Sampling the regularization parameter α

1. A residual principle:

$$\alpha_r : \| A_\Delta u^\alpha - f_\delta \|_{E_m} = \delta_{E_m}, \quad \left(\delta_{E_m} \gg h \| u^\circ \|\right), \quad \delta_{E_m} \simeq \left(\sum_{n=1}^{m} \sigma_n^2\right)^{\frac{1}{2}}.$$

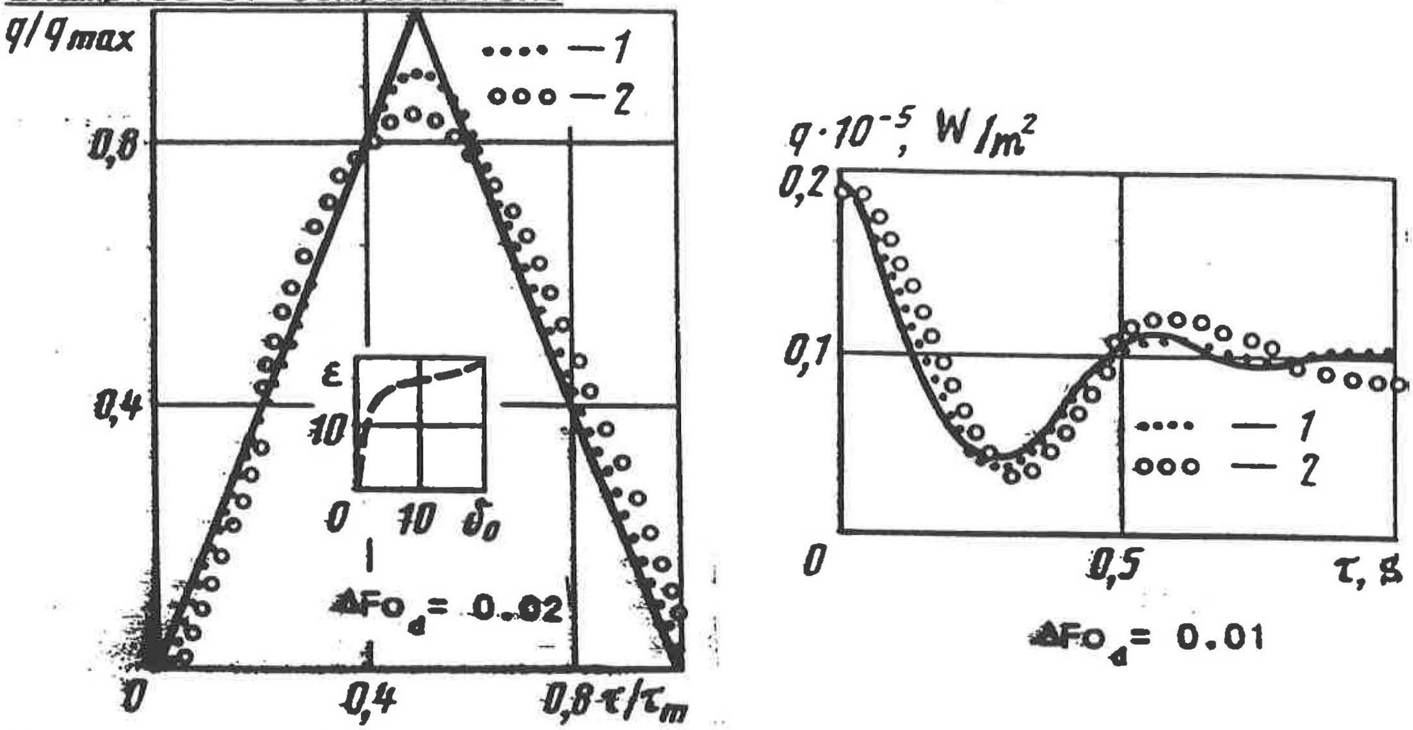
2. A method of quasioptimal parameter:

$$\min_\alpha \| \alpha \frac{du^\alpha}{d\alpha} \|$$

If $\alpha_{j+1} = k\alpha_j$, где $0 < k < 1$, $k \sim 1$, $\quad \alpha \frac{du^\alpha}{d\alpha} \simeq \frac{u^{\alpha_{j+1}} - u^{\alpha_j}}{k-1}$

$$\alpha : \| u^{\alpha_{j+1}} - u^{\alpha_j} \|$$

Examples of computations



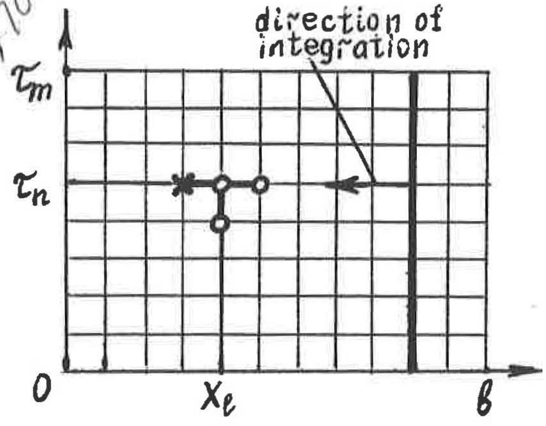
Normal distribution of errors in input temperatures; ——— is true solutions; .... is solutions of IHCP when $3\sigma = 0.01\ T^*_{max}$; •••• is solutions of IHCP when $3\sigma = 0.1\ T^*_{max}$.

REGULARIZATION OF NONLINEAR IHCP

Suggested by Alifanov in 1970s

1. By-interval of time regularization.
2. Regularized continuation of the solution of a heat conduction equation.

depend on choice of discretization scheme



System of regularizing functions:

$$\Phi_\alpha[T_l] = \|M_l T_l - g_l\|^2_{E_m} + \alpha_l \Omega_l,$$

where l is an index of the space levels;
M_l is a transition matrix from space layers $l+2, l+1$ to layer l;
$T_l = [T_{l1}, T_{l2}, \ldots, T_{lm}]^T$ is an unknown vector of temperatures in l th layer;
g_l is a vector of known parameters; α_l is a regularization parameter; Ω_l is a stabilization function, for example:

$$\Omega_l = k_1 \|\Delta T_l\|^2_{E_m} + k_2 \|\Delta^2 T_l\|^2_{E_m}, \quad k_1 \geq 0, \quad k_2 > 0,$$

first finite difference   second finite difference

where Δ and Δ^2 are the first and second finite differences, respectively.

$$\alpha=\alpha_r : \quad \rho_l(\alpha) \equiv \sum_{n=1}^{m} (T^\alpha_{Ln} - T^*_n)^2 = \delta^2$$

$$\delta^2 \simeq \sum_{n=1}^{m} \sigma_n^2$$

$$q(\tau) = -\lambda(T(0,\tau))\partial T(0,\tau)/\partial x$$

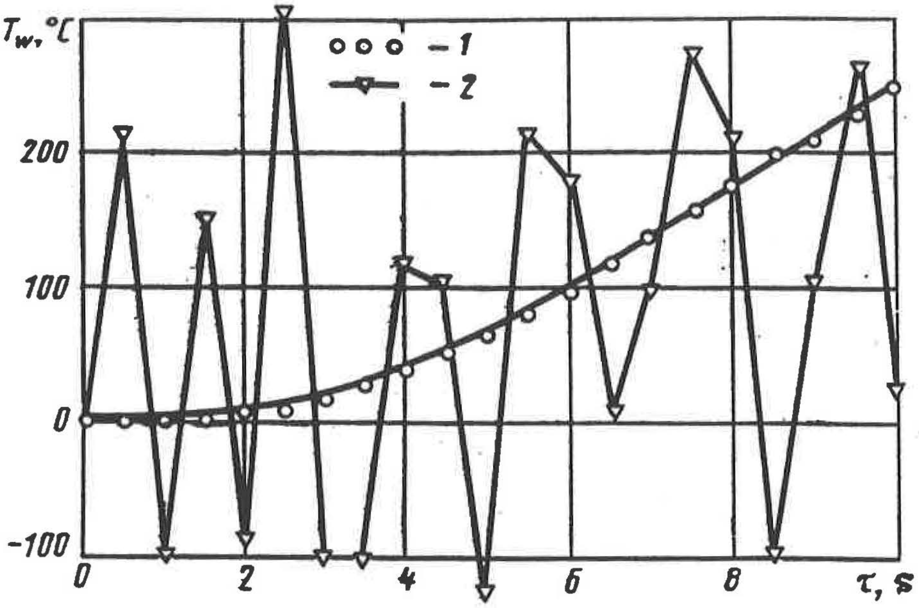
Same finite difference scheme must be used in forward solver and α solver?

Choice of regularization parameters α_l:

$$\alpha_l = \alpha_{l_{residual}} : \qquad \sum_{n=1}^{m} \left(T^{\alpha}_{Ln} - T^{*}_{n} \right)^2 = \delta^2_{E_m} ,$$

where T^{α}_{Ln} is temperature calculated at the point $x = b$ (from the solution of a direct problem);

T^{*}_{n} is the experimental temperature.



Example of computations: —— is an exact solution;

Normal distribution of errors in the input temperatures,

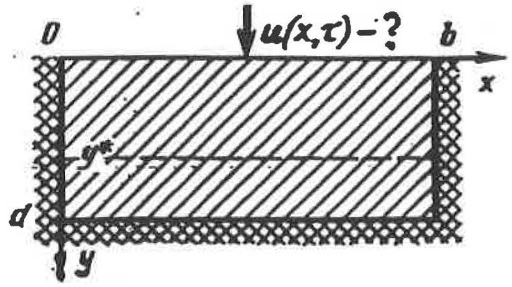
$3\sigma = 0.05 \cdot T^{*}_{max}$; $\lambda(T)$ and $C(T)$ corresponded to a graphite

$q^{*}(\tau) = 0$;

1 - the regularized numerical method;

2 - the direct numerical method ($\alpha_l = 0, \quad l = \overline{1, L}$) .

REGULARIZATION OF TWODIMENSIONAL IHCP



$$Au = \int_0^{\tau} d\xi \int_0^{b} u'(x\ ,\xi)K(x,x';\tau-\xi)dx' =$$

$$= f^{*}(x,\tau),\ \tau\in(0,\tau_m];\ x\in(0,b). \quad (\ 15\ )$$

Approximation: $A_{\Delta}u = f^{*}$. (16)

Regularizing functional:

using Green's functions

$$\Phi_{\alpha}[u] = \int_0^{\tau_m} d\tau \int_0^{b} \left(A\ u - f^{*}\right)^2 dx + \alpha \int_0^{\tau_m} d\tau \int_0^{b} \left(\frac{\partial^2 u}{\partial x\ \partial \tau}\right)^2 dx \quad (\ 17\ )$$

$\alpha>0$

Finite-difference analogy of $\Phi_{\alpha}[u]$:

$$\hat{\Phi}_{\alpha}[u] = \sum_{n=1}^{m}\ \sum_{l=1}^{L}\left(f_{ln} - f^{*}_{ln}\right)^2 + \alpha\sum_{n=1}^{m}\ \sum_{l=1}^{L}\left(u_{ln} - u_{l,n-1} - u_{l-1,n} + u_{l-1,n-1}\right)^2, \quad (\ 18\ )$$

natural boundary conditions

$\min\limits_{u} \hat{\Phi}_{\alpha}[u]$ under the conditions $u_{0n} = u_{1n},\ \ u_{L+1,n} = u_{Ln},$

$n = \overline{0,m+1};\ \ u_{l0} = u_{l1},\ \ u_{l,m+1} = u_{lm},\ \ l = \overline{0,L+1}$:

$$\boxed{A_{\Delta}^{T}\ A_{\Delta}\ u + \alpha\ C\ u = A_{\Delta}^{T}\ f^{*},} \quad (\ 19\ )$$

where

$$C = \begin{bmatrix} s & -s & & & \\ -s & p & -s & & \\ & -s & p & -s & \\ & & & \ddots & \\ & & -s & p & -s \\ & & & s & s \end{bmatrix}_{j\times j},\quad S = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}_{L\times L},\quad P = \begin{bmatrix} 2 & -2 & & & \\ -2 & 4 & -2 & & \\ & -2 & 4 & -2 & \\ & & & \ddots & \\ & & -2 & 4 & -2 \\ & & & -2 & 2 \end{bmatrix}_{L\times L}$$

where $j = L\times m$;

XIII.
Iterative Regularization Method

ITERATIVE REGULARIZATION

The iterative regularization method can be used for solving various inverse problems (retrospective, boundary, coefficien geometric and combined problems) both linear and nonlinear, i nonoverdetermined and overdetermined formulations.

Linear case

$$\boxed{Au = f}, \quad A : U \Rightarrow F; \qquad (20)$$

$A \subseteq \mathcal{L}(U, F)$; U, F are Hilbert spaces;

A^{-1} - unbounded or does not exist;

{ f, A } is precise initial data.

Assume that $U_f = \{ u \in U : Au = f \} \neq \emptyset$.

$$\Delta(u) = \|Au - f\|_F, \qquad J(u) = \frac{1}{2}\|Au - f\|_F^2 = \frac{1}{2}\Delta^2(u)$$

$$\boxed{J'u = A^*(Au - f)}; \quad \omega = -J'u$$

$$A^* : F \to U \quad (Au, f)_F = (u, A^* f)_U, \quad u \in D_A, f \in D_{A^*}$$

linear convergence

Steepest descent method

$$u_{n+1} = u_n - \beta_n J' u_n, \quad \beta_n = \| J' u_n \|_U^2 / \| A J' u_n \|_F^2 . \qquad (21)$$

Super-linear rate convergence

Conjugate gradient method

$$u_{n+1} = u_n - \beta_n p_n, \quad p_n = J' u_n + \gamma_{n-1} p_{n-1}, \quad p_0 = J' u_0, \qquad (22)$$

$$\beta_n = \left(J' u_n, p_n \right)_U / \| A p_n \|_F^2 ,$$

$$\gamma_{n-1} = \left(J' u_n, J' u_n - J' u_{n-1} \right) / \| J' u_{n-1} \|^2 ,$$

where $J' u_n = A^*(A u_n - f)$ - the gradient of residual functional.

Exact initial data

Steepest descent Simple iteration

conjugate gradient

V.M.Friedman (1962), W.J.Kammerer, M.Z.Nashed (1972):

$$\lim_{n \Rightarrow \infty} u_n = u^\circ, \quad u^\circ = \underset{u \in U_f}{\operatorname{argmin}} \| u - u_0 \| \text{ - normal solution}$$

initial estimate

finite dimensional operator

errors from measurements

Initial data with errors

$$A_h u = f_\delta, \quad u \in U, \quad f_\delta \in F, \qquad (23)$$

where $A_h \in \mathcal{L}(U, F)$,

$\{f_\delta, A_h\}$ is given approximation to initial data:

$\|A_h - A\| \leqslant h; \; f_\delta = f + \tilde{f}, \; \|\tilde{f}\|_F \leqslant \delta$

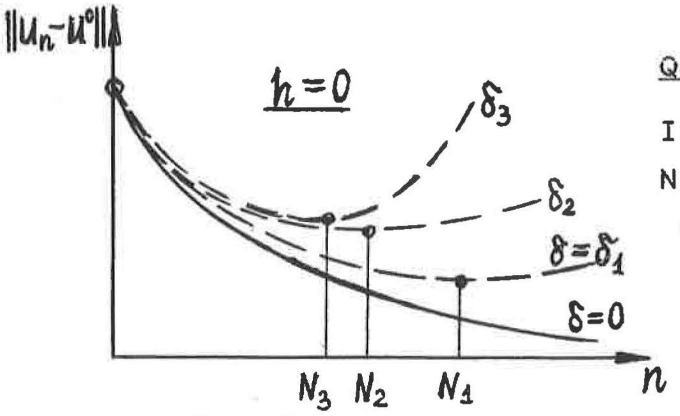
$\sigma = \{\delta, h\}$ is the initial data error

Case 1. The gradient methods for solving ( ) may be even divergent

Case 2. The gradient sequences converge to certain elements u_σ

$$\sigma = \{\delta, h\} \Rightarrow u_\sigma \qquad u_\sigma$$

$$u_\sigma \underset{\sigma\to 0}{\not\to} \tilde{u} \in U_f$$



$\delta_3 > \delta_2 > \delta_1$

Question:

It there any way of choosing $N(\sigma)$ which would provide $u_{N(\sigma)} \xrightarrow[\sigma\Rightarrow 0]{} u^0$?

JES !

1) converge with exact data necessary, but not sufficient

2)

Sufficient conditions for regularization of a certain iterative method

$$u_{n+1} = \Phi (u_n, \delta, h) . \qquad (24)$$

Theorem 1. If

1) for $\delta = h = 0$ $\exists \lim\limits_{n \Rightarrow \infty} u_n = \overline{u} \in U_f$ $(\overline{u} = \overline{u} (u_0))$;
2) $\Phi (u_n, \delta, h)$ is continuous at all points $\{ u, 0, 0 \}$ except maybe $\{ u, 0, 0 \}$: $u \in U_f$;
3) $\forall u \in U_f$, $\Phi (u, 0, 0) = u$.

Then $\exists N (\sigma)$: $\lim\limits_{\sigma \Rightarrow 0} u_{N(\sigma)} = \overline{u}$.

Theorem 2. Steepest descent method and conjugate gradient method satisfy conditions of Theorem 1.

Thus, appropriate regularized approximations converge to normal (with respect to initial approximation) solutions of Eq.(20) with σ error tending to zero.

HOW TO CHOOSE N (σ) ?

usually not known

Residual criterion:

Denote $\Delta_n = \| A_h u_n - f_\delta \|_F$, $\Delta_r = h \| u^\circ \| + \delta$.

Theorem 3. For the steepest descent method

$$\exists \quad N_r = \min_n \left\{ n : \frac{\Delta_n^2 + \Delta_{n+1}^2}{2\Delta_n} < C \Delta_r \right\}, \quad C > 1$$

and $\lim_{\sigma \Rightarrow 0} u_{N_r} = u^\circ$.

$$\| u_n - u^\circ \| \geq \| u_{n+1} - u^\circ \| , \quad \forall\, n < N_r .$$

useful when met

Similar criterion has also been obtained for the conjugate gradient method.

$h \| u^\circ \| << \delta$

Generalized Residual Criterion

If solution to (20) is unique, then u° in Δ_r may be replaced by u_n

$$\Delta_{rn} = h \| u_n \| + \delta$$

Nonlinear case

The validity of this approach for solution of ill-posed problems in nonlinear formulations has been demonstrated by the computer-experiment technique.

Frechet derivative a linear operator

$$Au = f, \quad u \in U, \; f \in F,$$

where $A: U \to F$ is a nonlinear continuous Frechet differentiable operator; U and F are Hilbert spaces

The gradient $J'u = (A'u)^*(Au - f)$,
where $A'u$ is the Frechet derivative of the operator A at the point u;
$(A'u)^*$ is the operator adjoint of the operator A'

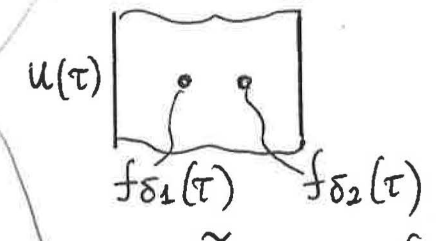
The steepest descent method:

$$u_{n+1} = u_n - \beta_n J'u_n, \quad n = 0, 1, \ldots,$$

$$\beta_n : J(u_{n+1}) = \min_{\beta \geq 0} J(u_n - \beta J'u_n)$$

① If the input data are first smoothed it is possible to use the traditional iteration-stopping methods (e.g. $\| u_{n+1} - u_n \|_U \leq \varepsilon$) .

The stopping by additional measurement



$u(\tau)$ is unknown

$$f\delta_1(\tau) = f_1(\tau) + \Delta f_1(\tau)$$
$$f\delta_2(\tau) = f_2(\tau) + \Delta f_2(\tau)$$

additive random errors

$$\tilde{N} : \min_n \left\{ \int_0^{\tau_m} \left[T(u_n(\tau), d_2, \tau) - f\delta_2(\tau) \right]^2 \right\}$$

② Determination of the gradient of the residual functional

$$J [u] = \frac{1}{2} || A u - f ||_F^2 \Longrightarrow J' u .$$

Two analytical methods:

- with help Green's functions (for linear problems);
- on the base of adjoint problems (both for linear and nonlinear problems); $(A'u)^*$

An account of a priori information about a solution:

- qualitative information, such as, smoothness of unknown functions;
- quantitative information about the values of the functions and their derivatives.

Two methods:

1) A direction of discent is chosen in the initial space L_2 or R^n, but in such a way that obtained approximations remain in the class of smooth functions.
2) The iterative sequence is constructed directly in Sobolev's space W_2^1 .

Modifications of gradient algorithms (for solution of multiparameter inverse problems)

Two cases:

1) To determine simultaneously a certain set of functions and parameters (e.g. several coefficients in equation with partial derivatives, two boundary conditions, boundary and initial conditions, and so on).

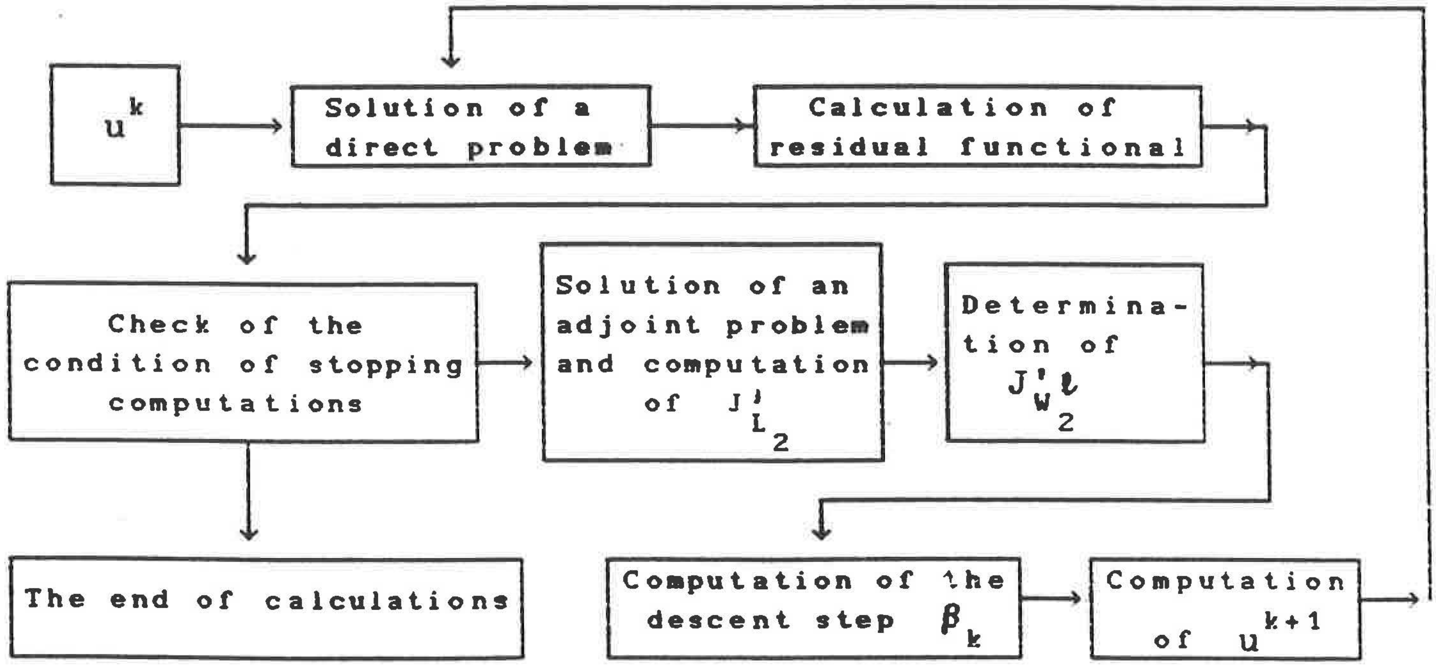
2) To consider smoothness of unknown function when the function is represented in terms of one of its derivatives and in terms of values at specific points.

The gradient-method modifications: instead of the scalar quantity of descent step the descent-step vector is determinec at each iteration for each individual component on the basis of residual minimization.

Stopping the iteration, when initial data error is unknown

1. Stopping by additional measurements
2. Stopping by the increment of functional

A general flow-chart of the computational algorithm



Coefficient Inverse Heat Conduction Problem

$$C(T)T_\tau = (\lambda(T)T_x)_x, \quad (x,\tau) \in Q = (0,b) \times (0,\tau_m];$$

$$T(x,0) = \xi_0, \quad x \in [0,b];$$

$$T(0,\tau) = T_1(\tau), \quad T(b,\tau) = T_2(\tau), \quad \tau \in [0,\tau_m];$$

$$T(d_i,\tau) = f_i(\tau), \quad i = \overline{1,N}, \quad \tau \in [0,\tau_m],$$

$$0 < d_1 < \ldots < d_N < b$$

only one measurement
essential for
uniqueness –
but more
is
better

where $C(T)$; $T_1(\tau)$; $T_2(\tau)$; $f_i(\tau)$, $i = \overline{1,N}$ are known functions; ξ_0; b; τ_m; d_i, $i = \overline{1,N}$ are given numbers

$$\lambda(T) - ?$$

$\lambda(T), \; T \in [T_0, T_M]$

$$\lambda(T): \min_{\lambda(T)} | J(\lambda) - \delta_J^2 |,$$

$$J(\lambda) = \sum_{i=1}^{N} \int_0^{\tau_m} \rho_i(\tau) \overbrace{[T(\lambda, d_i, \tau) - f_i(\tau)]^2}^{\text{residual}} d\tau$$

weights for different
measurements

Approximation of $\lambda(T)$ by cubic B-splines

Uniform grid for $[T_0, T_M]$:

$$\{T_j = T_0 + jH\},\ j = \overline{0, M}, \qquad \text{step} \qquad H = \frac{T_M - T_0}{M}$$

$$T_{-3} = T_0 - 3H,\ T_{-2} = T_0 - 2H,\ T_{-1} = T_0 - H;$$

$$T_{M+1} = T_0 + .(M + 1)H,\ T_{M+2} = T_0 + (M + 2)H;\ T_{M+3} = T_0 + (M + 3)H$$

Cubic B-splines: $\{B_j\}_{-1}^{M+1}$

$$B_j(\bar{T}) = B_0(\bar{T} - jH),\quad j = -1, 0, 1, \ldots, M + 1;$$

$$B_0(\bar{T}) = \frac{1}{6H^3}[(\bar{T} - 2H)_+^3 - 4(\bar{T} - H)_+^3 + 6(\bar{T})_+^3 - 4(\bar{T} + H)_+^3 +$$

$$+ (\bar{T} + 2H)_+^3];$$

where $\bar{T} = T - T_0$; $(\bar{T} - g)_+^3 = \begin{cases} (\bar{T} - g)^3 & \text{при } \bar{T} \geqslant g \\ 0 & \text{при } \bar{T} < g. \end{cases}$

$$\tilde{\lambda}(T) = \sum_{j=-1}^{M+1} \lambda_j B_j(\bar{T}),$$

where λ_j are unknown parameters

Variational formulation:

$$\lambda = [\lambda_{-1}, \lambda_0, \dots, \lambda_{M+1}]^T : \min_{\lambda} | \hat{J}[\lambda] - \delta_J^2 |,$$

(L7) *L

where $\hat{J}(\lambda) = \sum_{i=1}^{N} \int_0^{\tau_m} \rho_i(\tau) [T(\tilde{\lambda}, d_i, \tau) - f_i(\tau)]^2 \, d\tau$

Iterative algorithm

$$\lambda^{k+1} = \lambda^{\bar{k}} - \beta_k \mathbf{p}^k, \quad k = 0, 1, \dots, \bar{k},$$

$$\boxed{\bar{k} : \hat{J}(\lambda^k) \simeq \delta_J^2 ;} \ \beta_k : \min_{\beta} \hat{J}(\lambda^k - \beta \mathbf{p}^k),$$

residual condition

Can use same method for direct problem and adjoint problem

where $\mathbf{p}^k = \vec{\mathbf{J}}'(\lambda^k) + \gamma_k \mathbf{p}^{k-1},$

$$\gamma_0 = 0; \quad \gamma_k = \frac{\sum_{j=-1}^{M+1} J'^{k}_{\lambda_j} (J'^{k}_{\lambda_j} - J'^{k-1}_{\lambda_j})}{\sum_{j=-1}^{M+1} (J'^{k}_{\lambda_j})^2}, \quad k > 0$$

The gradient components: $\mathbf{J}'(\lambda)$

adjoint variable

$$J'_{\lambda_j} = \sum_{i=1}^{N+1} \int_0^{\tau_m} d\tau \int_{d_{i-1}}^{d_i} \frac{\psi_i(x, \tau)}{C(x, \tau)} [T_{xx}(x, \tau) B_j(\bar{T}) +$$

$$+ T_x^2(x, \tau) B_j'(\bar{T})] \, dx$$

$$j = -1, 0, \dots, M+1,$$

where $\psi(x,\tau)$ is adjoint variable

Descent step:

from linear estimator ↓

$$\beta_k = \frac{\sum_{i=1}^{N} \int_0^{\tau m} [T(\lambda^k, d_i, \tau) - f_i(\tau)]\, v(\Delta\tilde{\lambda}^k, d_i, \tau)\, d\tau}{\sum_{i=1}^{N} \int_0^{\tau m} v^2(\Delta\tilde{\lambda}^k, d_i, \tau)\, d\tau}$$

where $v(\Delta\tilde{\lambda}^k, d_i, \tau)$ is the temperature increment:

$$v_\tau = a_1 v_{xx} + a_2 v_x + a_3 v + \frac{1}{C}(\Delta\tilde{\lambda}^k T_{xx} + \Delta\tilde{\lambda}^k_T T_x^2),$$

$(x, \tau) \in Q$;

$v(x, 0) = 0$;

$v(0, \tau) = v(b, \tau) = 0$;

$$\Delta\tilde{\lambda}^k = \sum_{j=-1}^{M+1} P_j^k B_j(\bar{T}).$$

Solution sequence (each iteration)

Step 0 → data L, ρC_p, T_0 $\overbrace{T_1, T_L}^{\text{boundary}}$ temps

(T_i, ρ_i), $\rho_i \simeq 1$ (can be)

Need some $\lambda_0(T) = \lambda^* =$ const (maybe)

Step 1 → $k = k+1$ iteration # increment

- Solve direct problem using current λ
- Calculate residual, if $\leq \delta^2$ → done

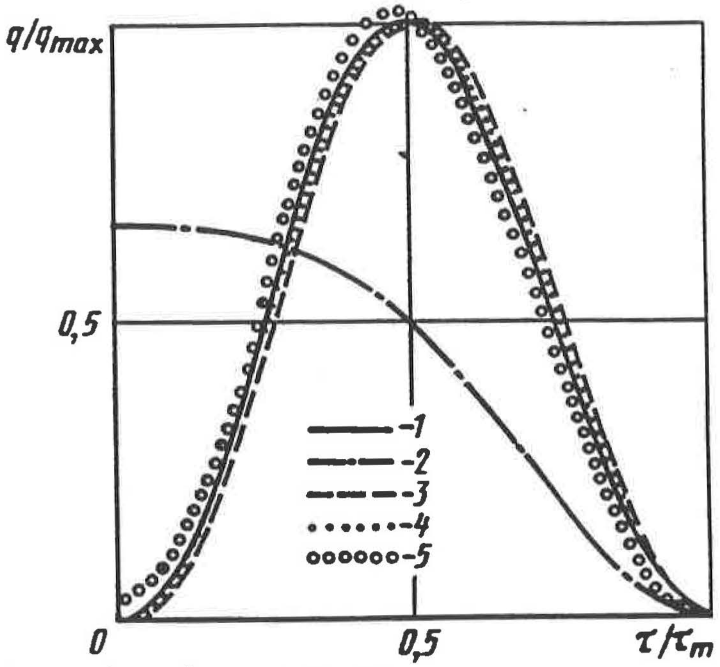
Step 2 Solve adjoint problem for $\psi(x,t)$ requires $T(d_i,\tau) - f_i(\tau)$ from direct problem and data.

Step 3 if $k \geq 1$ compute γ_k $(\gamma_0 = 0)$

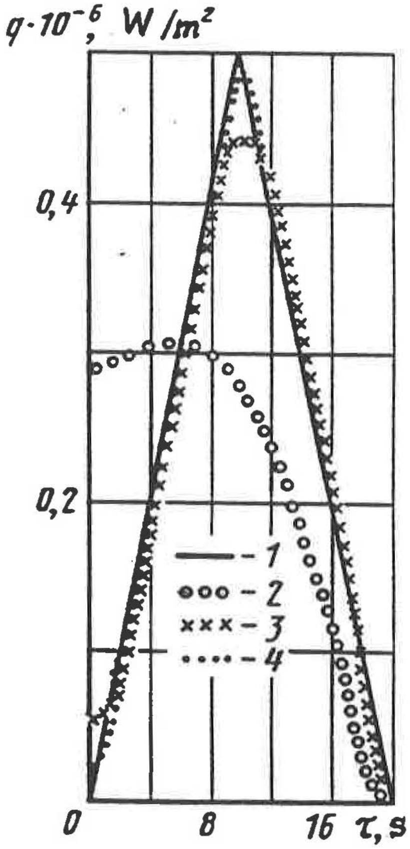
Results of calculations

I. Boundary IHCP

Linear cases and non-disturbed data (spase L_2)



b = d, $\Delta Fo_b = 0.01$



b = 0.003 m, d = 0.002 m
$\Delta Fo_b = 0.015$

1 - a true solution

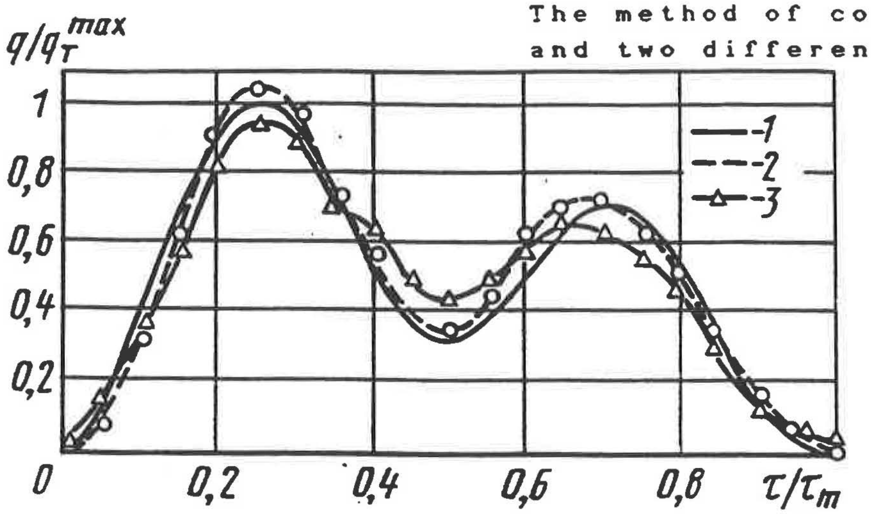
2, 3, 4 - the 1st, 9th and 50th approximations (conjugate-gradients); 5 - 50th approximation (steepest descent)

2, 3, 4 - the 1st, 3d and 100th approximations (conjugate gradients)

Δt is less than Δt_{crit} - "step regularization" effect was not present

Nonlinear cases and disturbed data (space L_2)

The method of conjugate gradients
and two different methods of stopping



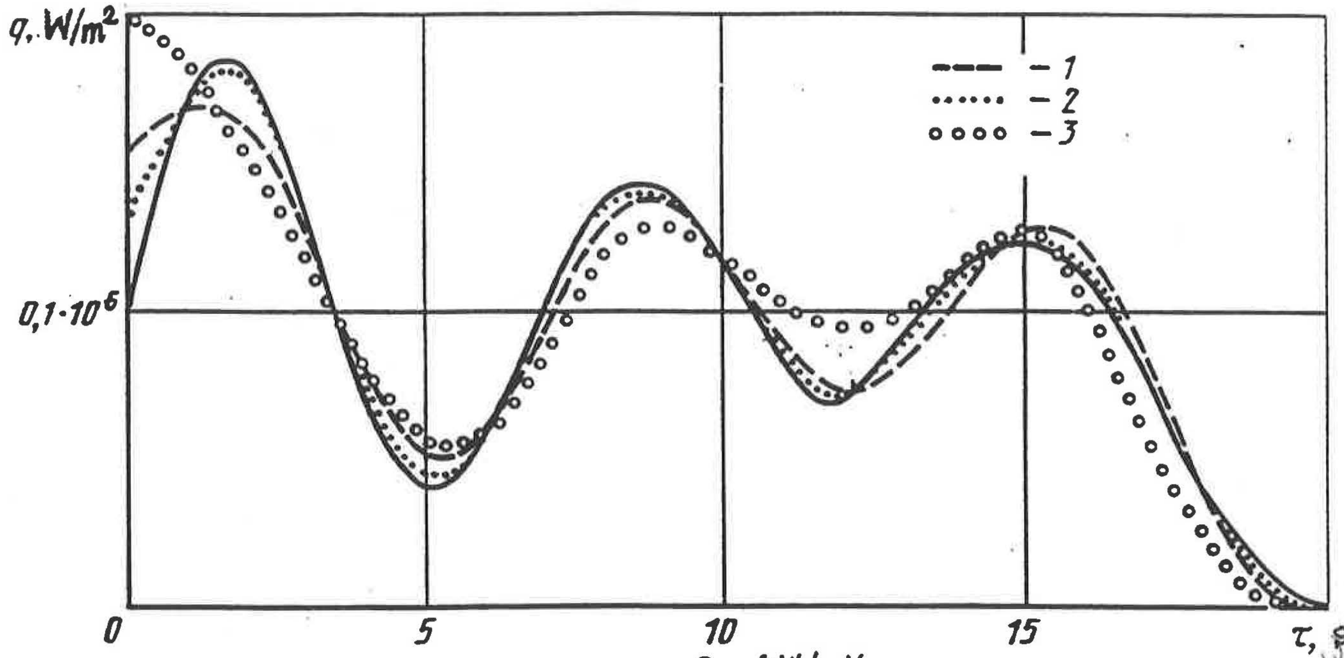
Normal law with dispersion of errors, $3\sigma = 0.05\ T^*_{max}$;
$d = 0.42\ b$, $0.06 \le \Delta Fo_d \le 0.1$

1 - the unknown function; 2 - stopping by the additional measurement at point $d' = 0.08 \cdot b$ ($N = 10$); 3 - stopping by the criterion of residual ($N = 7$)

properties used
were
asbestos
reinforced
plastic

data had
5% of max T
random error

Smoothed data



$b = 0.003$m; $\lambda = 0.721 + 0.288 \cdot 10^{-3} T$, kW/mK ;

$a = 0.4 \cdot 10^{-6} - 0.143 \cdot 10^{-9} T + 0.408 \cdot 10^{-12} T^2$, m²/s

(full curve - true solution)

1 and 2 - 30th and 60th approximations for non-disturbed data; 3 - the 60th approximation on smoothed data (initial values of temperature included errors distributed by normal law with dispersion $3\sigma = 0.05 \cdot T^*_{max}$)

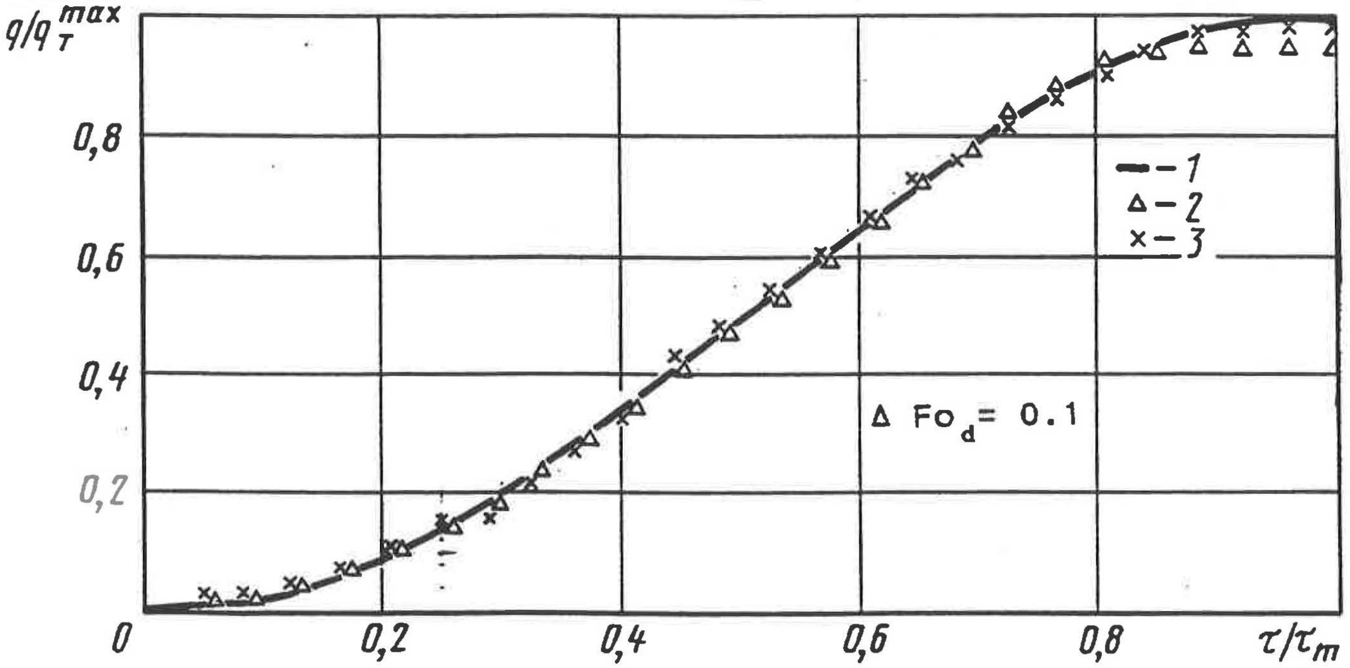
since data were presmoothed!

No instability
can use
usual
methods for
stopping

preliminary smoothing
with 2nd order regularization

Sobolew Space

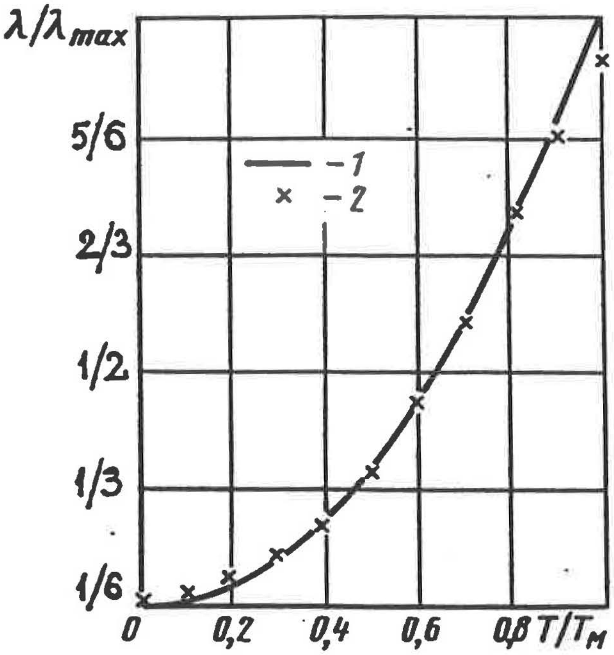
Solution of the boundary IHCP in space W_2^1 (an account of smoothness of the unknown function)



1 - a true solution; 2 - an approximation for disturbed input data by normal law, $3\sigma = 0.05 \cdot T^*_{max}$ (stopping by criterion of residual, N = 7); 3 - an approximation for exact input data, N = 50

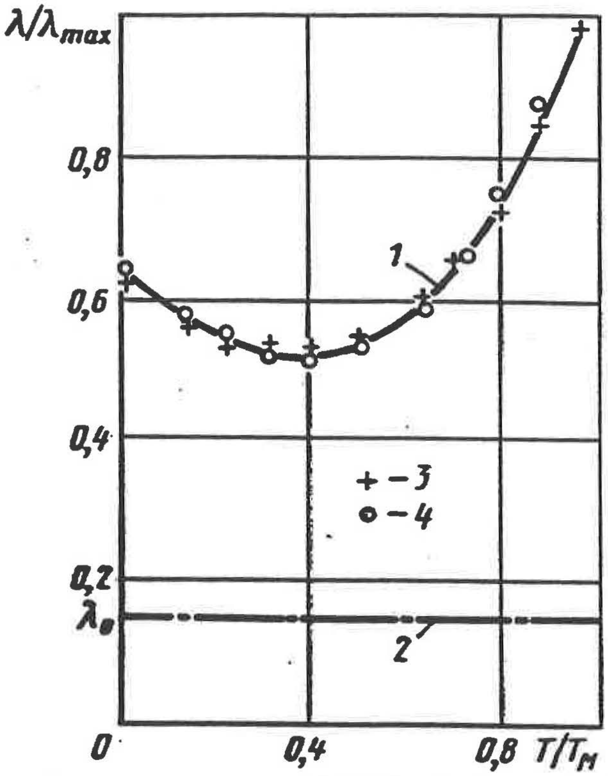
Cubic splines and conjugate gradient

II. Nonlinear coefficient IHCP - reconstruction of the thermal conductivity λ(T) with using spline-approximation



Exact input data

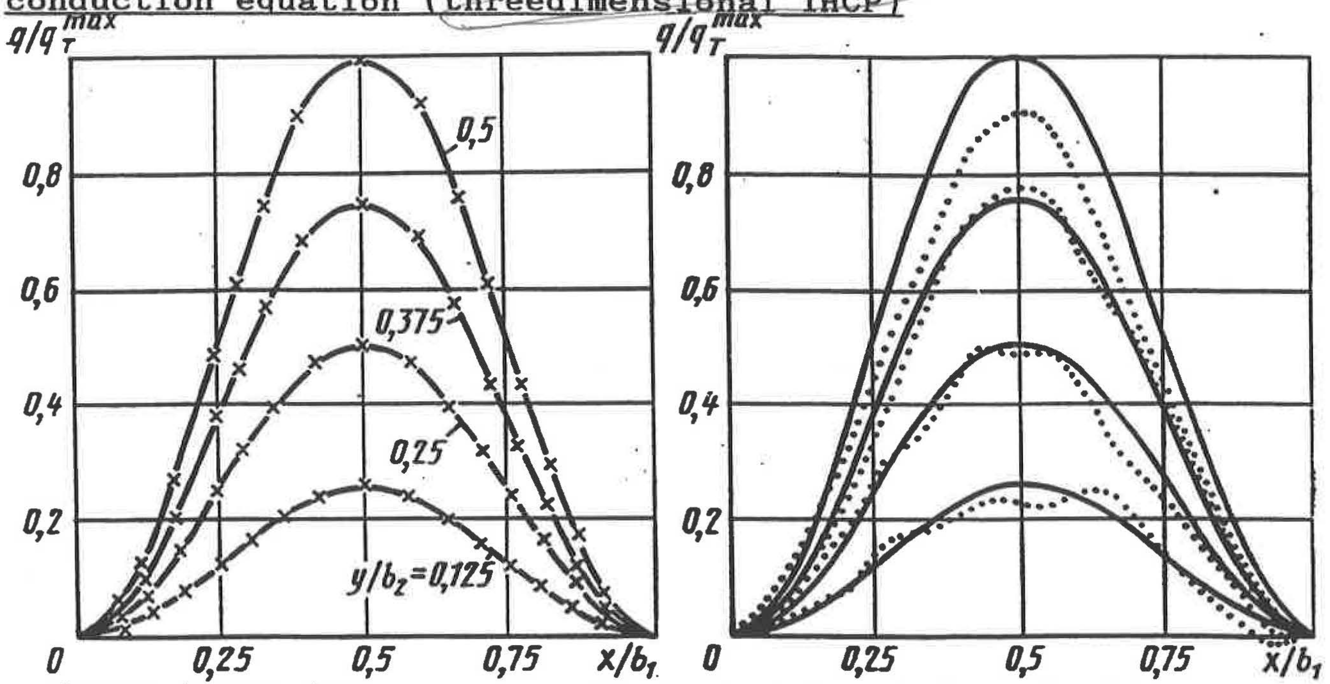
1 - the unknown function;
2 - the solution of IHCP.



Exact and disturbed input data

1-the unknown function; 2-the initial approximation; 3,4-the solution of IHCP for exact and disturbed data respectively.

III. The reconstraction of a source $Q(x, y, \tau)$ in the heat conduction equation (threedimensional IHCP)



Exact input data

Disturbed input data (uniform distribution of errors with maximum deviation 0.01 T^*_{max}

Results of computation for four cuttings by coordinate y and one time point $\tau = \tau_m / 2$ . .

XIV.

Comparison of Methods for Solving Boundary IHCPs

fast, can be used even for real time

Direct analytic and Semi-analytic

Most universal

Table

The conditions of effective application of the methods for the solution of boundary IHCPs

MN	Distinctions of application of methods	Direct approximate-analytical method (2)	Direct numerical method (3)	Iterative regularizatioin method (5)	Regularized algebraic method (4)	Regularized numerical method (4)
1	2	3	4	5	6	7
	<u>The peculiarities of a formulation of IHCP</u>					
1.	Linear with constant thermal properties	yes	yes	yes	yes	yes
2.	Nonlinear of linear with variable TP	no	yes	yes	no	yes
3.	Homogeneous heat-conduction equation	yes	yes	yes	yes	yes
4.	Generalized heat-conduction equation	no	yes	yes	no	yes
5.	Fixed bounderies of a body	yes	yes	yes	yes	yes
6.	Moving boundaries of a body	in onedimensional case	yes	yes	in onedimensional case	yes
7.	Fixed temperature gauses	yes	yes	yes	yes	yes
8.	Travelling temperature gauses	no	yes	yes	no	yes
9.	Onedimensional	yes	yes	yes	yes	yes
10.	Twodimensional	for the domains of simple shape	yes	yes	for the domains of simple shape	yes
11.	Overspecified	no	no	yes	no	no

General conditions of application					
12. The determination of heat loads	yes	yes	yes	yes	yes
13. The computation of temperature fields	no	yes	yes	no	yes
14. The slowly alterating heat-exchange processes (HP)	yes	yes	yes	yes	yes
15. The quickly alterating and short-term HP	no	no	yes	yes	yes
16. Low-temperature HP	yes	yes	yes	yes	yes
17. High-temperature HP with essentially variable intensity	no	yes	yes	no	yes

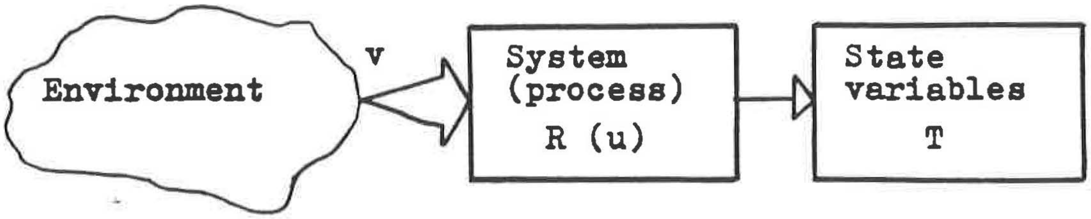
Continuation of table

1	2	3	4	5	6	7
18.	Small heat-depth of the instalation of temperature gauges	yes	yes	yes	yes	yes
19.	Big heat-depth of the installation of temperature gauges	no	no	yes	yes	yes
20.	Metallic structural elements	yes	yes	yes	yes	yes
21.	Thermal protection and thermal insulation	no	for materials with high heat conductivity only	yes	yes	yes
	The conditions of practical application					
22.	The complexity of the algorithms	simple	medium complexity		heightened complexity	complex
23.	Computation time expenses	small	small	medium	medium	large
24.	Computation in real time scale	possible				impossible
25.	The restrictions on the values of the steps of approximation	very strict	strict	no restriction		
26.	Preliminary smoothing of initial data for high levels of erros	necessary		not necessary		
27.	Obtaining of the needed degree of smoothness of the results	impossible		possible		
28.	An account of the quantitative a priori information about an unknown solution.	impossible		possible	hard to implement	

XV.

Development and Validation of Mathematical Models of Heat Transfer

Mathematical model

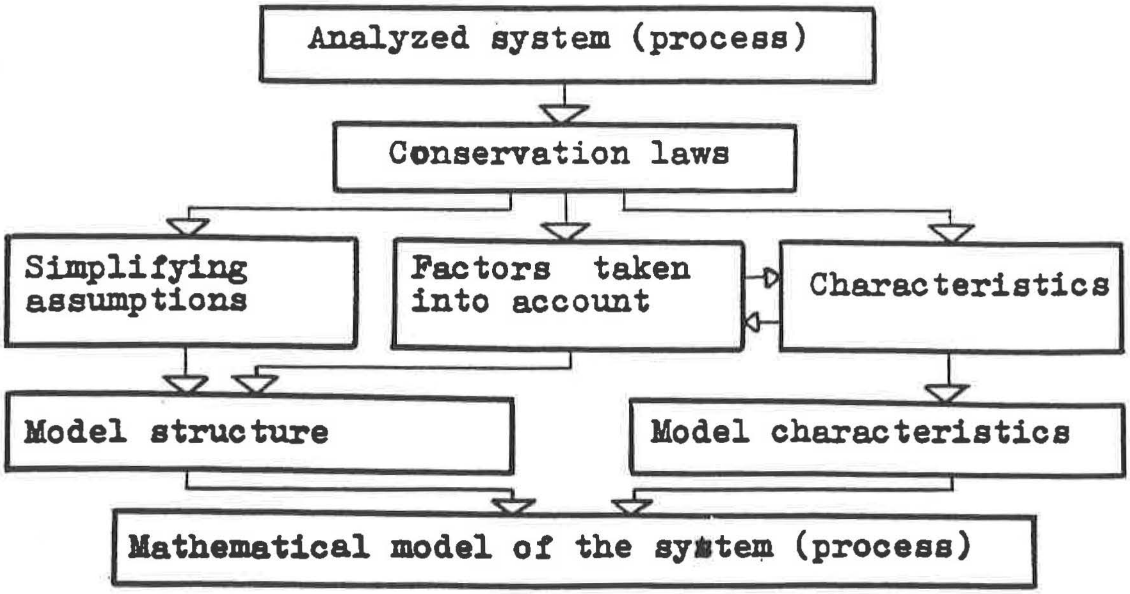


v - external (input) action;
u - vector of system characteristics;
$R(u)$- operator of the system;
T - state (output) variable.

Mathematical model - transformation of the external actions into state variables

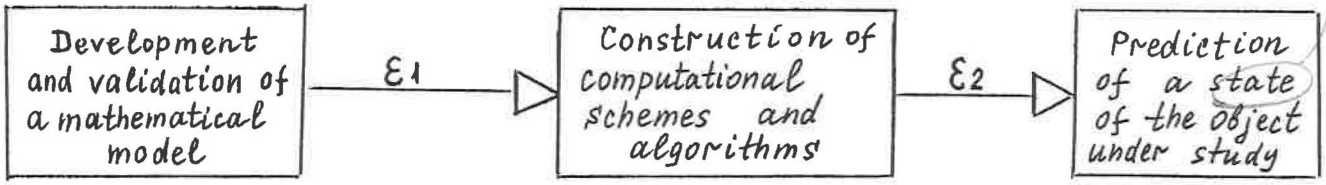
$$R\ (u)\ v = T$$

Mathematical model development



usually thermal state

Stages of Mathematical Modeling



Development and validation of a mathematical model

ε_1

Construction of computational schemes and algorithms

ε_2

Prediction of a state of the object under study

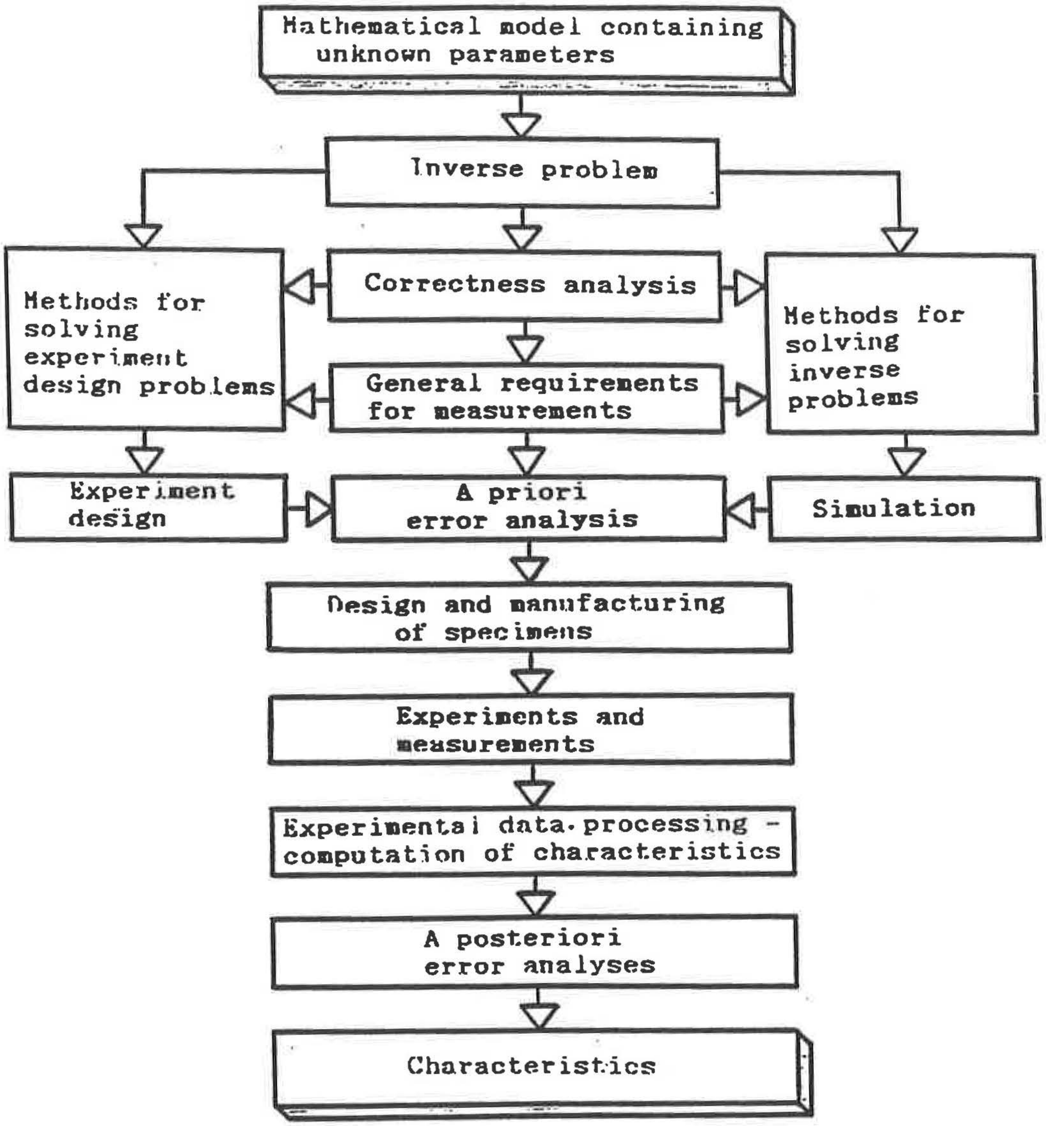
$\varepsilon_1, \varepsilon_2$ are errors of the modeling and calculations respectively

$$\varepsilon_2 = f(\varepsilon_1)$$

Development and validation of mathematical models:

- construction of a model structure;
- parametric identification;
- model validation for adequacy

Identification of characteristics



Mathematical model containing unknown parameters

Inverse problem

Methods for solving experiment design problems

Correctness analysis

Methods for solving inverse problems

General requirements for measurements

Experiment design

A priori error analysis

Simulation

Design and manufacturing of specimens

Experiments and measurements

Experimental data processing - computation of characteristics

A posteriori error analyses

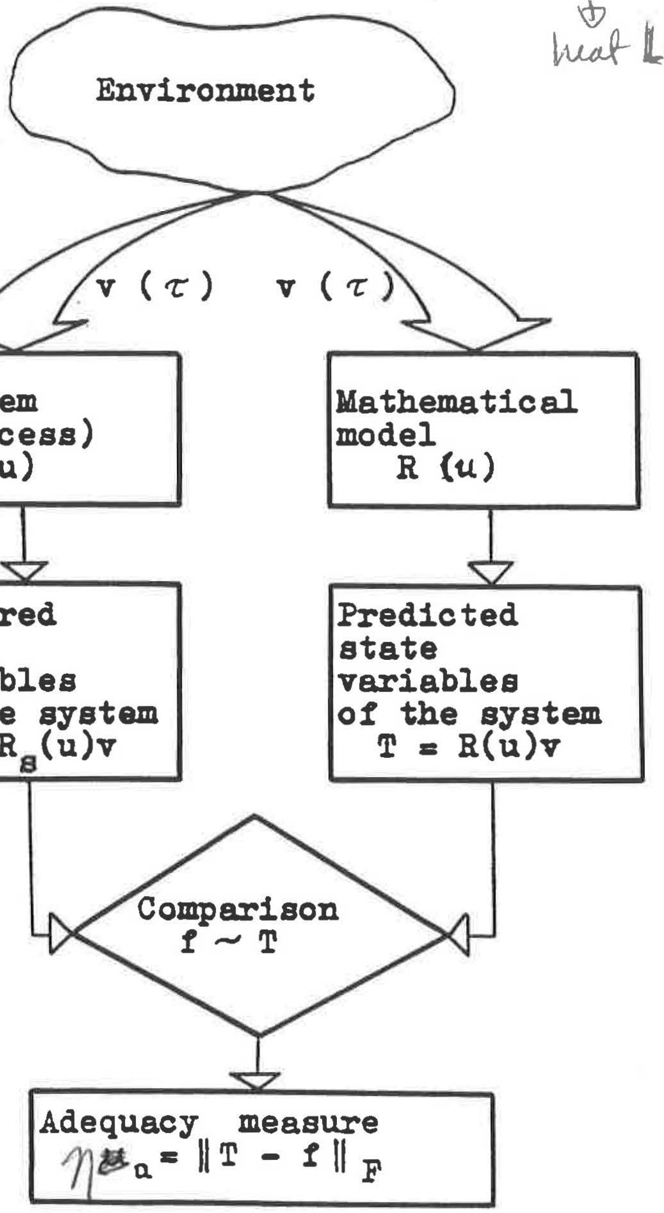
Characteristics

Examination of the model for adequacy

$$\eta(R) = \sup_{v \in V} \inf_{u \in U} \| \underbrace{R(u)} v - f \|_F$$

where U is a solution space; V is a set of potential loading actions

heat Loads



- **The Problem:** $\eta \leq \varepsilon$,

  $\varepsilon > 0$ - prescribed level of adequacy

Adequacy Criterion of Mathematical Model

Adequacy measure : $\eta = \eta(A,f) = \sup\limits_{f \in F_0} \inf\limits_{u \in U} \| Au - f \|_F$,

where $F_0 \subseteq F$ - some set of possible states of the process

For inexact input data : $\eta_\delta = \eta(A, f_\delta)$

Adequacy criterion : $\eta_\delta \leqslant \varepsilon(\eta, \max\limits_{f \in F_0} \delta_f)$,

where $\delta_f = \| f_\delta - f \|_F$ - error of some $f \in F_0$

Residual $\Delta = \| z - f_\delta \|_F$, $f_\delta = f_\delta(x,\tau)$, $z = z(x,\tau) = Au$

model reaction to change in desired quantity u

I. $F = L_2$:

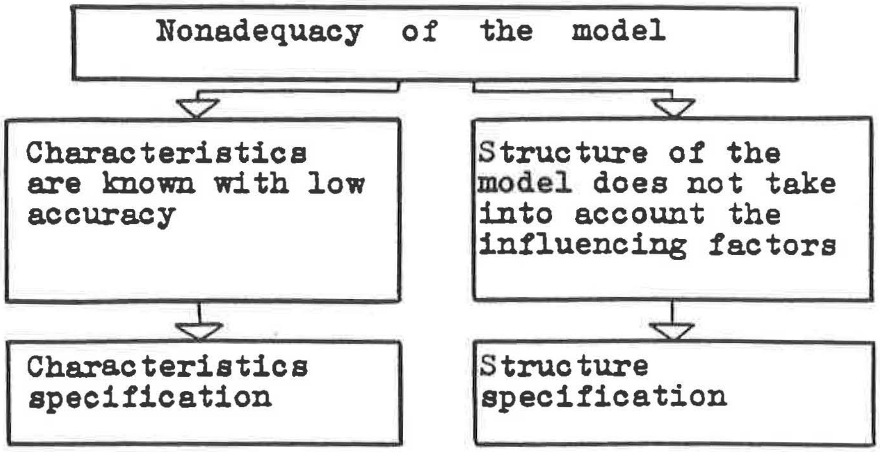
* if f is n - dimensional vector-function : $\Delta = \left\{ \int_0^{\tau_m} \int_\Omega [z - f_\delta]^T [z - f_\delta] d\Omega \, d\tau \right\}^{1/2}$

* if f is scalar field : $\Delta = \left\{ \int_0^{\tau_m} \int_\Omega [z - f_\delta]^2 d\Omega \, d\tau \right\}^{1/2}$

* if $f = f(\tau)$, $z = z(\tau)$: $\Delta = \left\{ \int_0^{\tau_m} (z - f_\delta)^T W (z - f_\delta) d\tau \right\}^{1/2}$, W is weight matrix

II. $F = C$: $\Delta = \max\limits_{x,\tau} | z - f_\delta |$

Reasons of nonadequacy of the model and ways of their elimination



Nonadequacy of the model

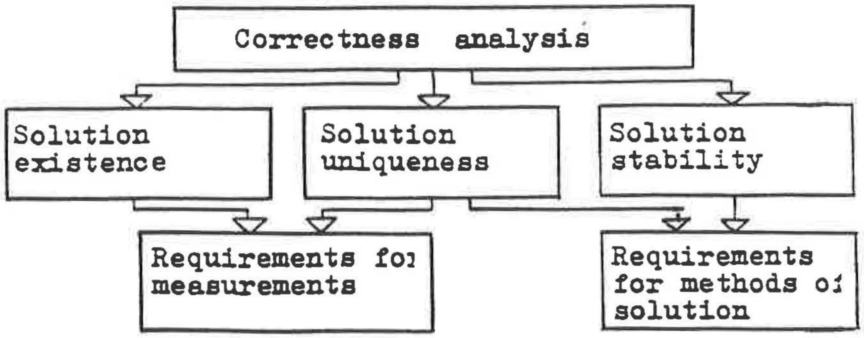
Characteristics are known with low accuracy

Structure of the model does not take into account the influencing factors

Characteristics specification

Structure specification

Correctness analysis of inverse problems



Correctness analysis

Solution existence

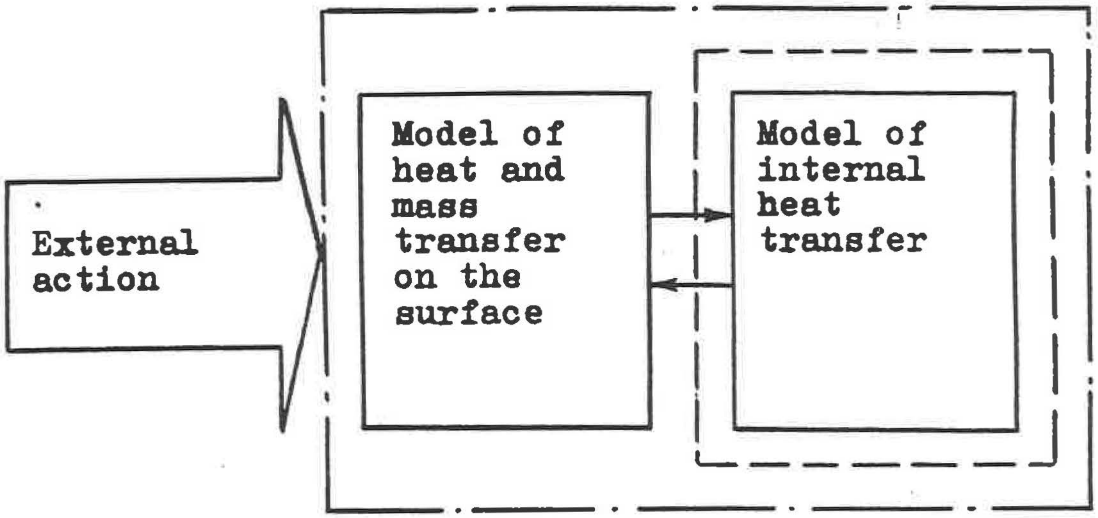
Solution uniqueness

Solution stability

Requirements for measurements

Requirements for methods of solution

General structure of a thermal mathematical model



External action

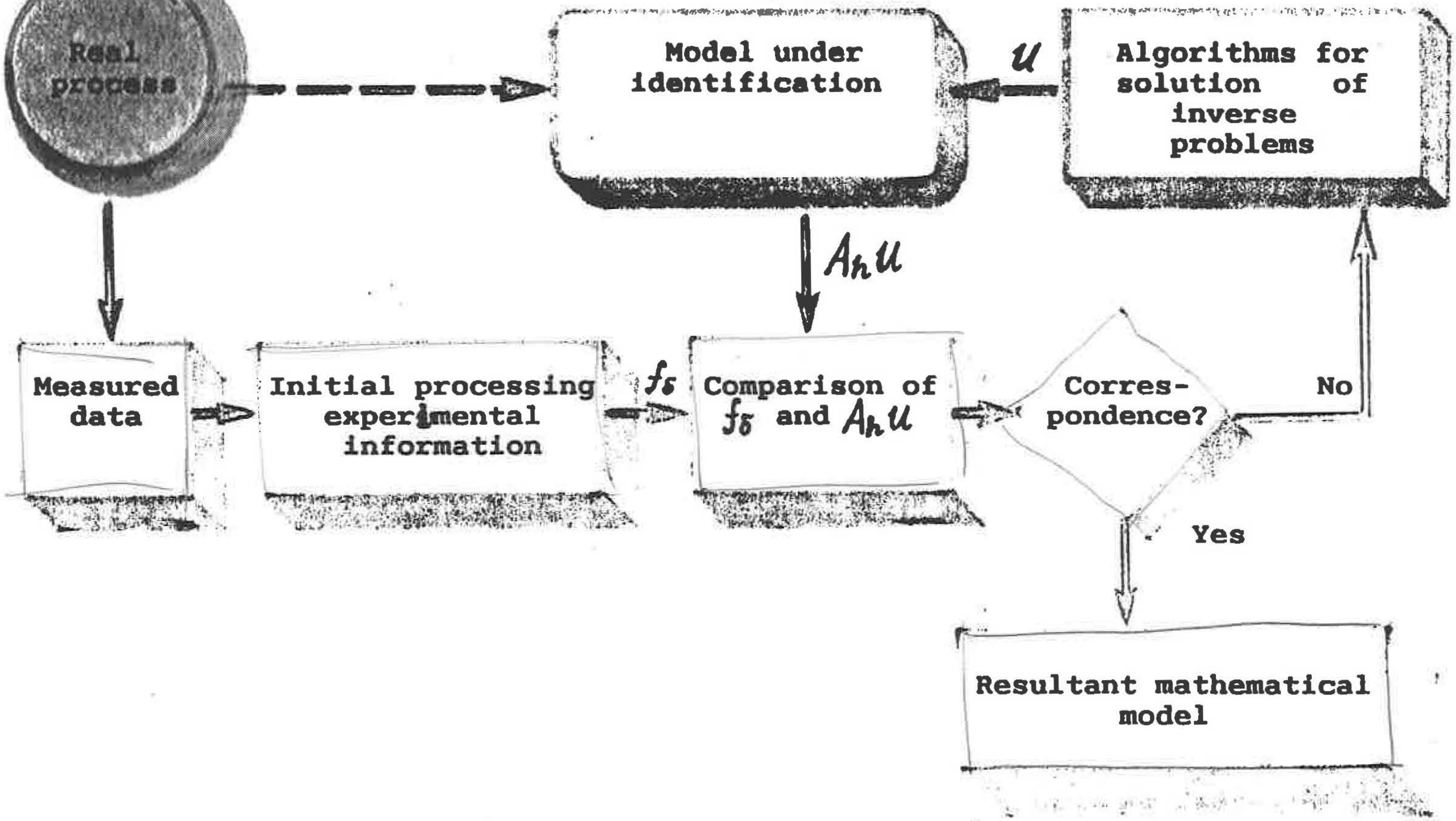
Model of heat and mass transfer on the surface

Model of internal heat transfer

Parametric Identification

$$A_h u = f_\delta$$

constant or vector



Real process

Model under identification

u

Algorithms for solution of inverse problems

$A_h u$

Measured data

Initial processing experimental information

f_δ

Comparison of f_δ and $A_h u$

Corres- pondence?

No

Yes

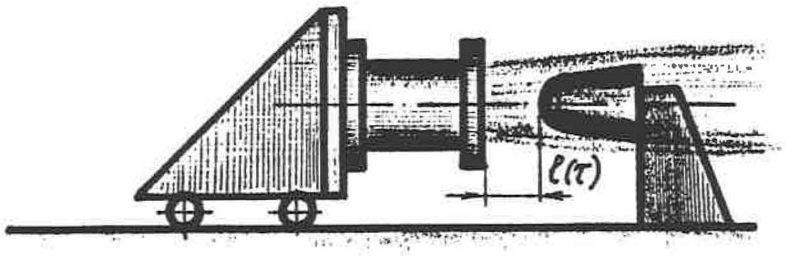
Resultant mathematical model

XVI.
Application Results

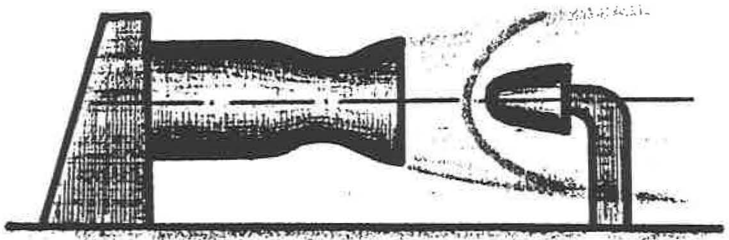
Practical Application Results (AR) of IHTP methods

AR1: **Non-Stationary Heat Measuring**

High - Temperature Gas Jets



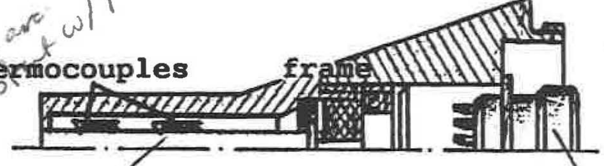
Plasma jets



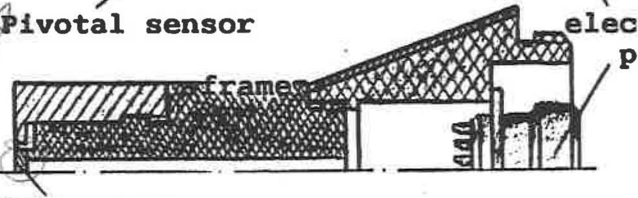
Jets of rocket engines

at least
2 required

Transient Calorimeters



Pivotal sensor

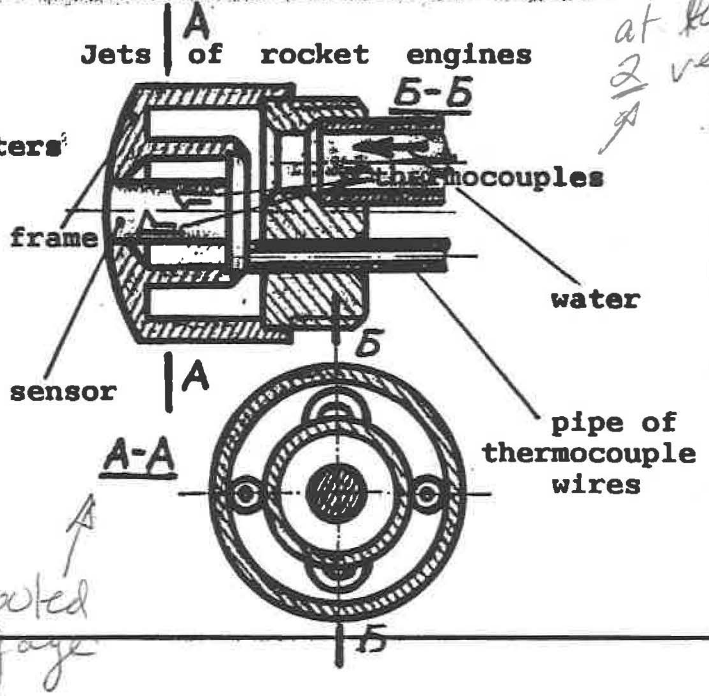


Disk sensor

Cu high conductivity
can operate to high T
Ni) thermal diffusivities are
Vanadium) nearly constant w/T

1-3mm

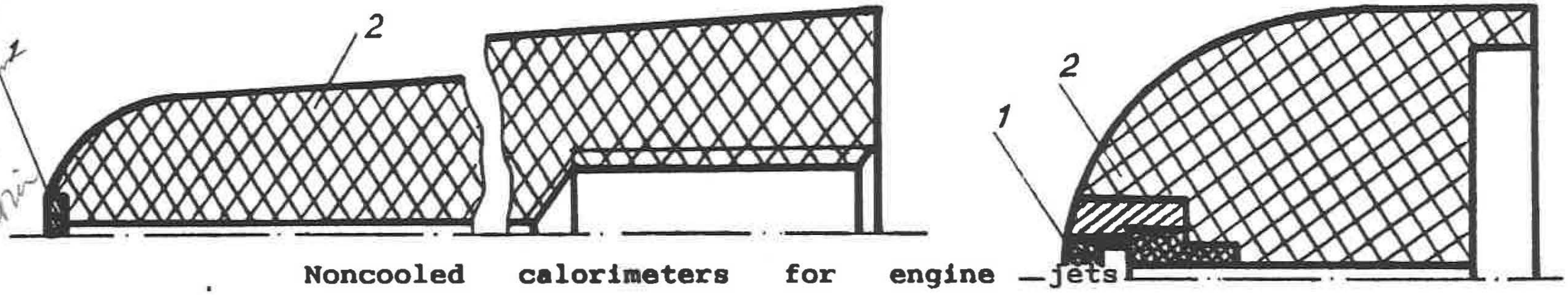
uncooled
gages



Cooled
gage

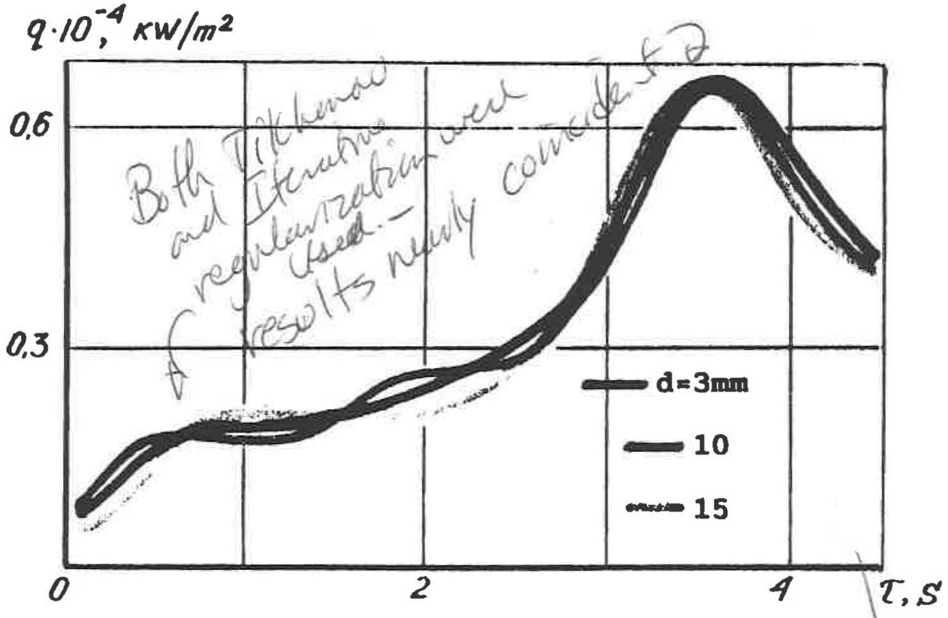
Accuracy of inverse method decreases with increasing distance of first sensor from active surface

redundant Temperatures - more than those needed for q estimation



Noncooled calorimeters for engine jets

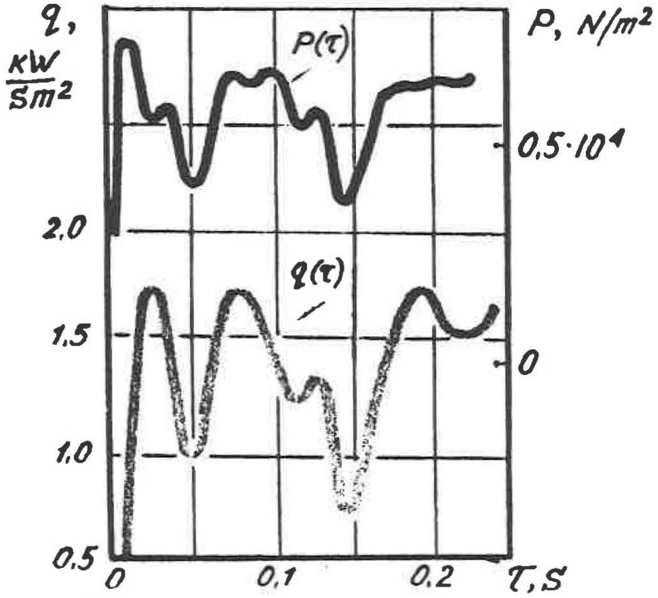
(1 - sensor, 2 - frame, 3,4 - plugs)



Both Tikhonov and Iterative regularization were used - results nearly coincide

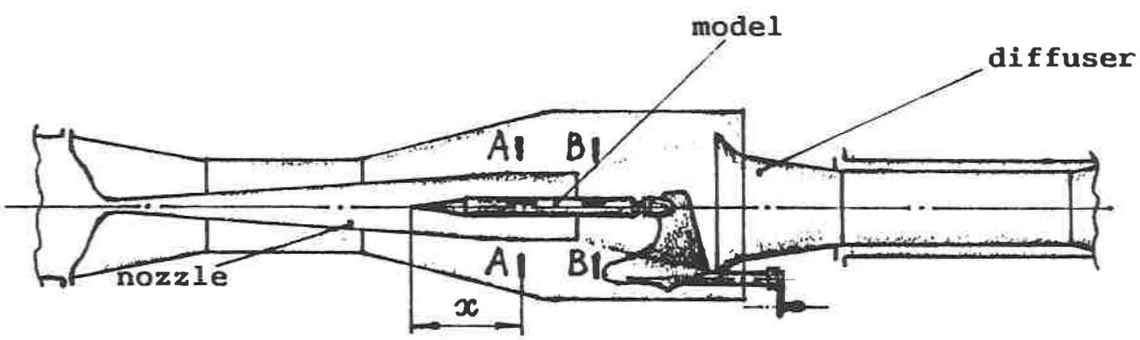
Results of determining of heat flux by pivotal calorimeter, moving along subsonic plasma jet

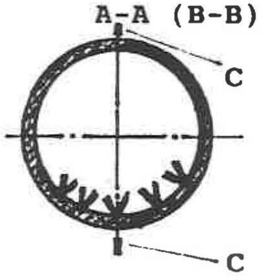
T data from longer than 4s were used -

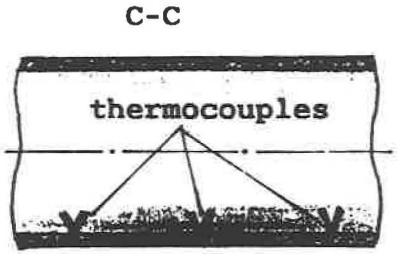


Heat fluxes in nitrogen plasma jet

AR2: Simulation of Transitional and Turbulent Heat Transfer Conditions





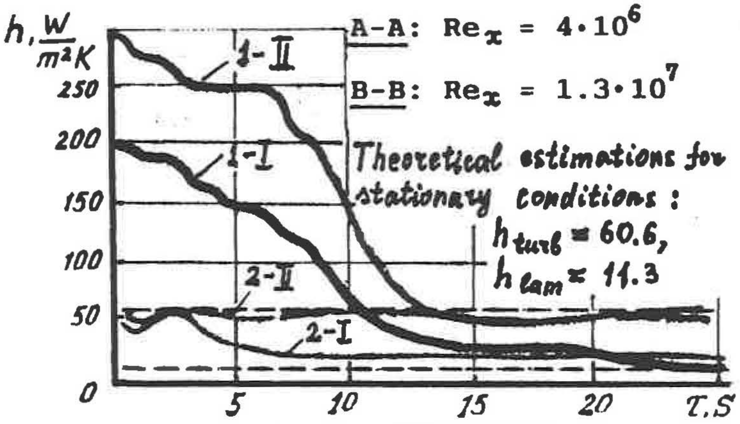


Supersonic thermal wind tunnel

Testing conditions: T_∞ = 150...500 °C; $p_0 = 8\cdot10^5$ Pa; M = 5.0

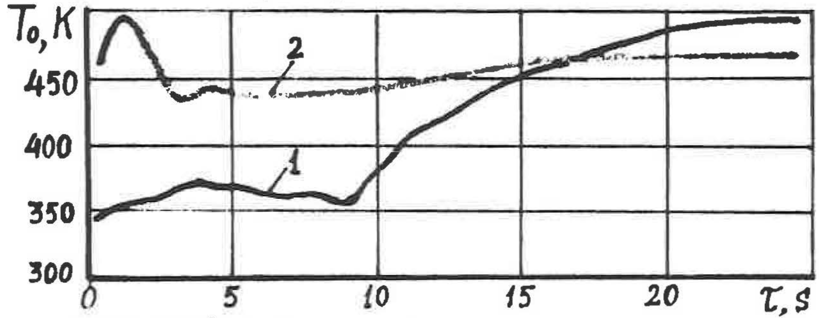
A-A: $Re_x = 4\cdot10^6$ - laminar boundary layer

B-B: $Re_x = 1.3\cdot10^7$ - turbulent boundary layer



Theoretical estimations for stationary conditions: $h_{turb} = 60.6$, $h_{lam} = 11.3$

Heat transfer coefficients



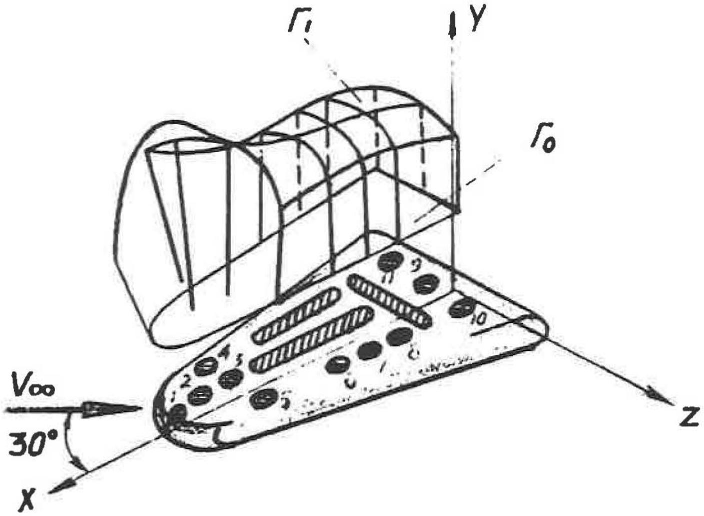
Stagnation temperature

(1,2 - numbers of tests; I - x = 0.3 m ; II - x = 1.0 m)

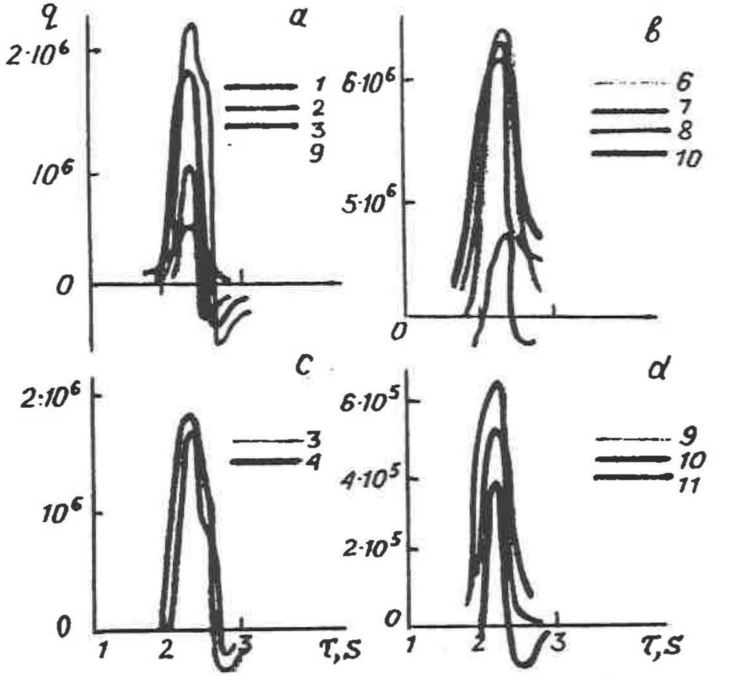
TP 1	AR3: Heat Transfer Diagnostics on Models Surface during testing at hypersonic wind tunnel with magnetogasdynamic accelarating air stream	3/4

● - one-dimensional sensors

▨ - two-dimensional sensors

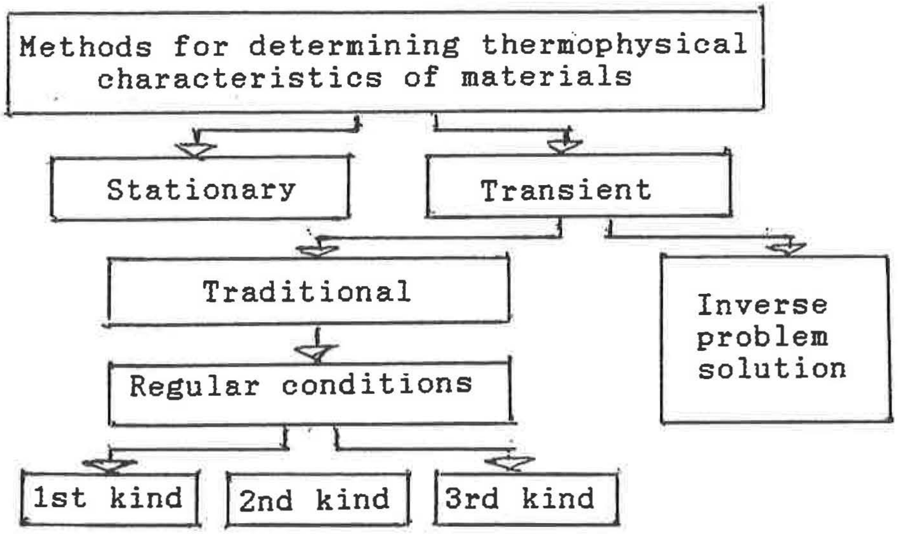


Arrangement of sensors and spacial heat flux distribution on the windward model surface: Γ_0 , original base surface; Γ_1 , flux distribution surface for the time moment τ = 2.3 s,
M = 20...25, Re $\simeq 10^5$, $\tau_{impulse} \simeq$ 1 s



Reconstructed values of heat flux q (τ), W/m^2 at diagnostics points in different model cross-sections:
a, at points 1,2,3,9;
b, at points 6,7,8,10;
c, at points 3,4;
d, at points 9,10,11.

Methods for characteristics determination



Traditional methods of characteristics

- Traditional methods are based on analytical solutions of the boundary-value problem for the heat conduction equation with constant coefficients.
- Heating conditions for samples used are rather simple and easy to reproduce.
- Rather narrow temperature range is realized in samples to provide constancy of characteristics.

Mathematical Model № 1
(fixed thermosensors and body boundaries)

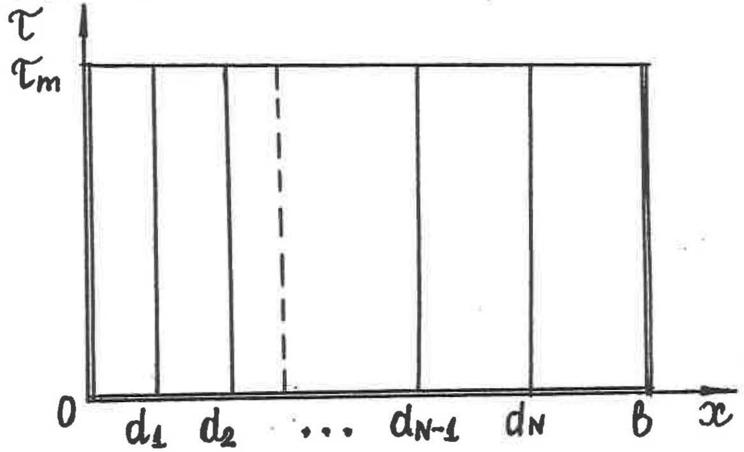
$$C(T)T_\tau = (\lambda(T)T_x)_x, \quad (x,\tau) \in Q = (0,b)\times(0,\tau_m);$$

$$T(x,0) = \xi(x), \quad x \in [0,b];$$

$$T(0,\tau) = T_1(\tau); \; T(b,\tau) = T_2(\tau), \quad \tau \in [0,\tau_m];$$

$$T(d_i,\tau) = f_i(\tau), \quad i = \overline{1,N}, \quad \tau \in [0,\tau_m];$$

$$C(T) > 0, \quad \lambda(T) > 0$$



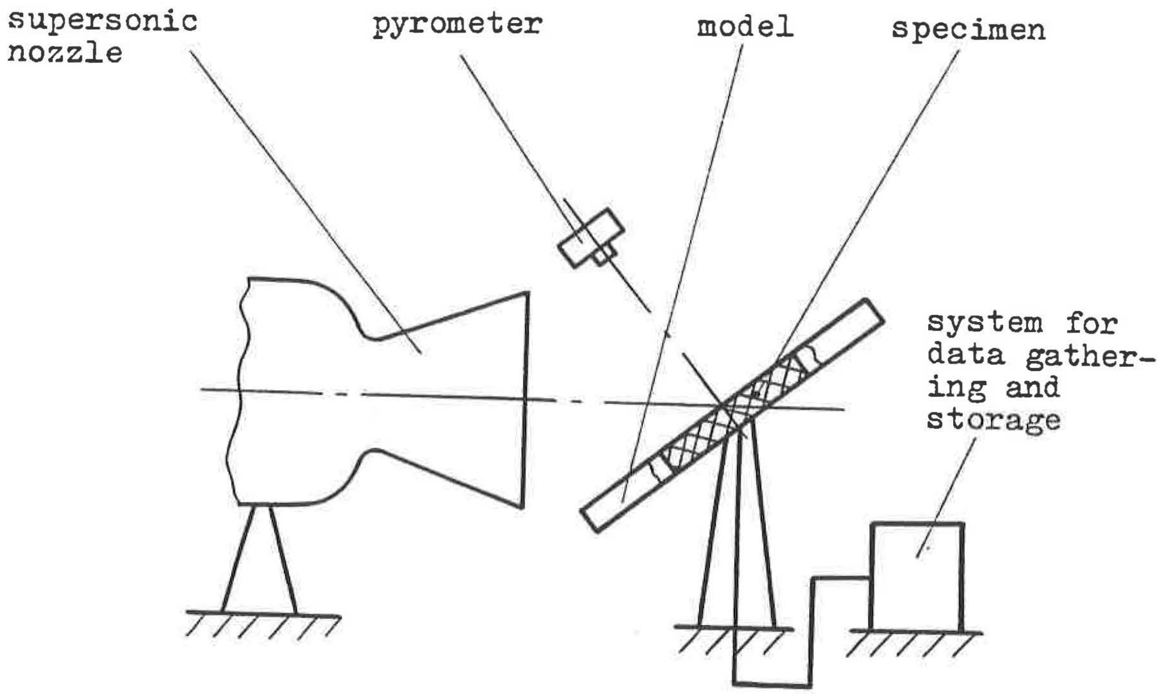
$$0 < d_1 < \ldots < d_N < b$$

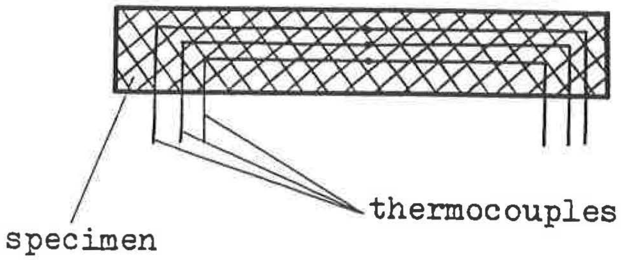
$$\lambda(T): \min_{\lambda(T)} |J(\lambda) - \delta_J^2|,$$

$$J(\lambda) = \sum_{i=1}^{N} \int_0^{\tau_m} \rho_i(\tau)\,[T(\lambda, d_i, \tau) - f_i(\tau)]^2 d\tau;$$

δ_J^2 is a minimal allowable level for residual

Diagram of test facilities





Results of temperature measurements

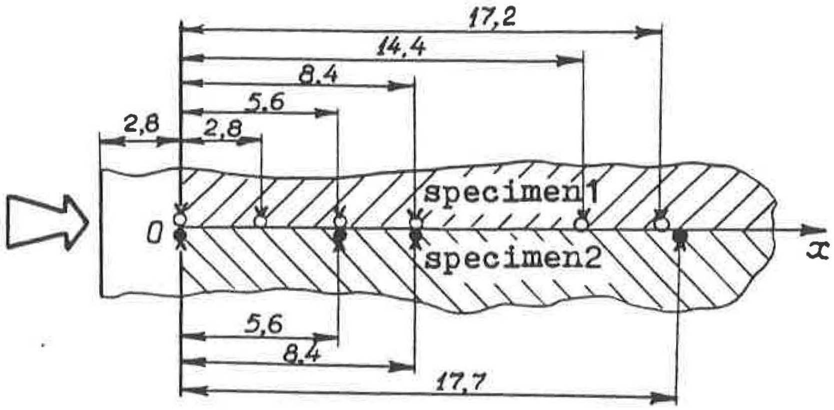
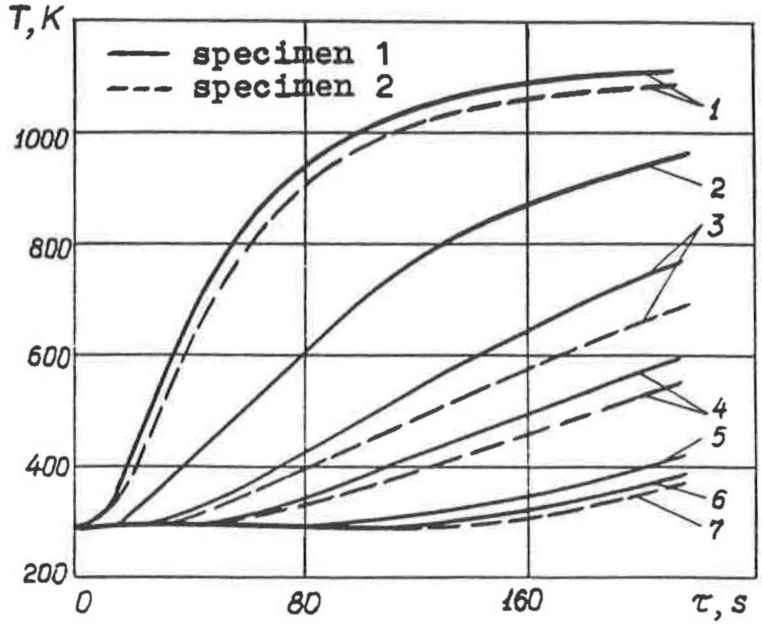


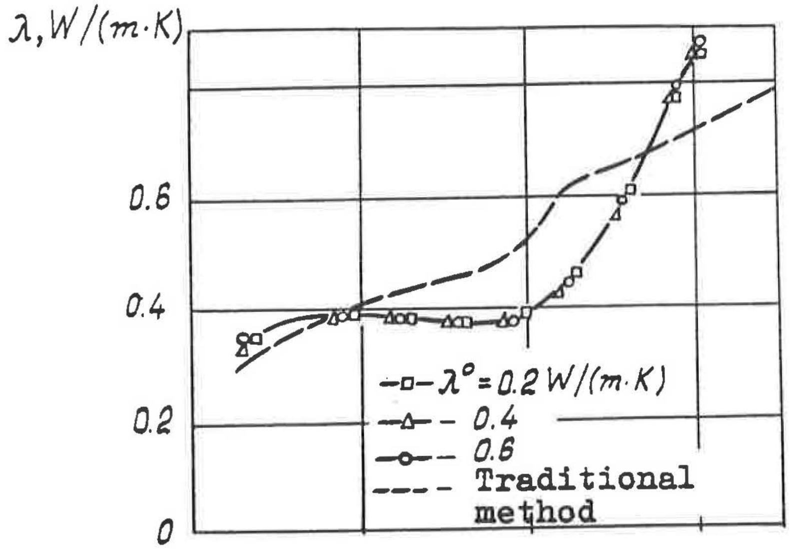
Diagram of thermocouples location.



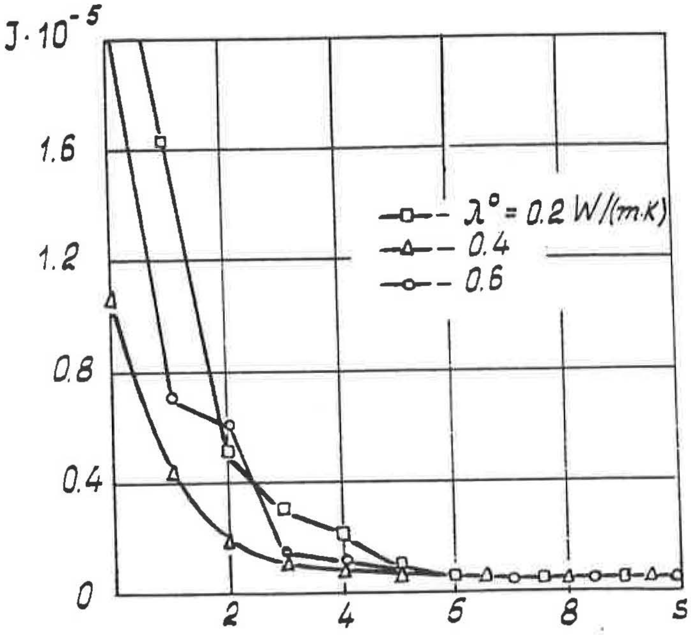
Measured temperatures:

1 - X=0; 2 - X=2.8 mm; 3 - X=5.6 mm; 4 - X=8.4 mm;
5 - X=14.4 mm; 6 - X=17.2 mm; 7 - X=17.7 mm.

Results of experimental data processing, specimen 1

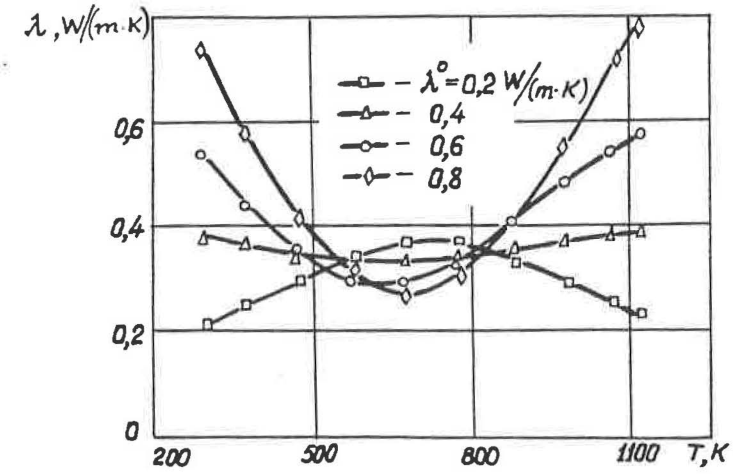


Thermal conductivity $\lambda(T)$ determination by solving the inverse problem with different values of initial approximation λ°.

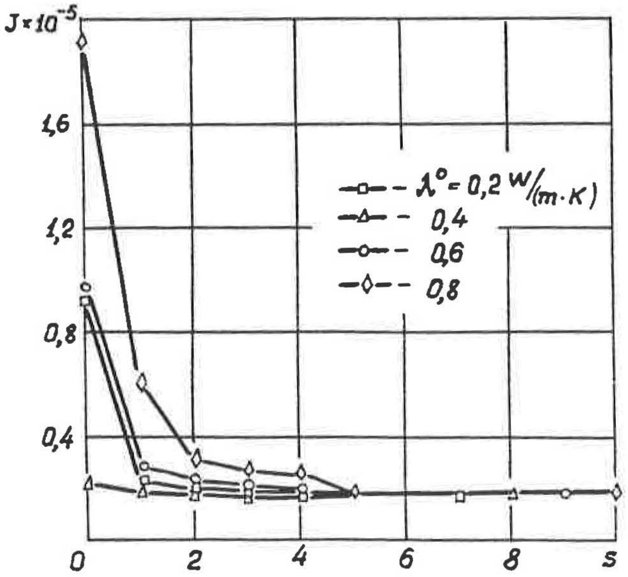


Changing residual functional J depending on iteration number s.

Results of experimental data processing, specimen 2



Thermal conductivity $\lambda(T)$ determination by solving the inverse problem with different values of initial approximation λ°.



Changing residual functional J depending on iteration number s .

Optimal Measurement Design

Measurement design $\varepsilon = \{N, \mathbf{d}\}$,
where N is the number of thermosensors

$\mathbf{d}$ is the vector of sensor coordinates($\mathbf{d} = \{d_i\}_1^N$

Problem: $\varepsilon = \varepsilon^*: \quad \max\limits_{\varepsilon} \Psi[F(\varepsilon)]$,

where $F(\varepsilon) = \frac{1}{N} \Phi_{jn}, \quad j, n = \overline{1, M}$;

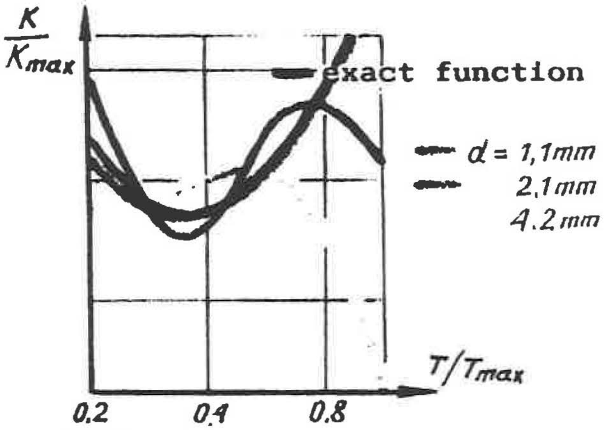
$M + 3$ - the number of intervals for $\lambda(T)$ approximation by B-splines

$$\Phi_{jn} = \sum_{i=1}^{N} \int_0^{\tau_m} \theta_j(d_i, \tau)\, \theta_n(d_i, \tau)\, d\tau ;$$

$\theta_j(d_i, \tau)$ are the sensitivity functions,

$$\theta_j(d_i, \tau) = \frac{\partial T(d_i, \tau)}{\partial \lambda_j}, \quad j = -\overline{1, M+1}$$

Experiment Design

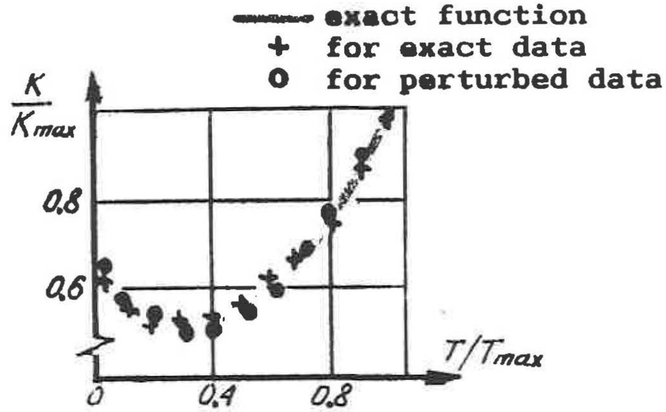


K(T) reconstruction using exact input data for different values at the thermocouple coordinate

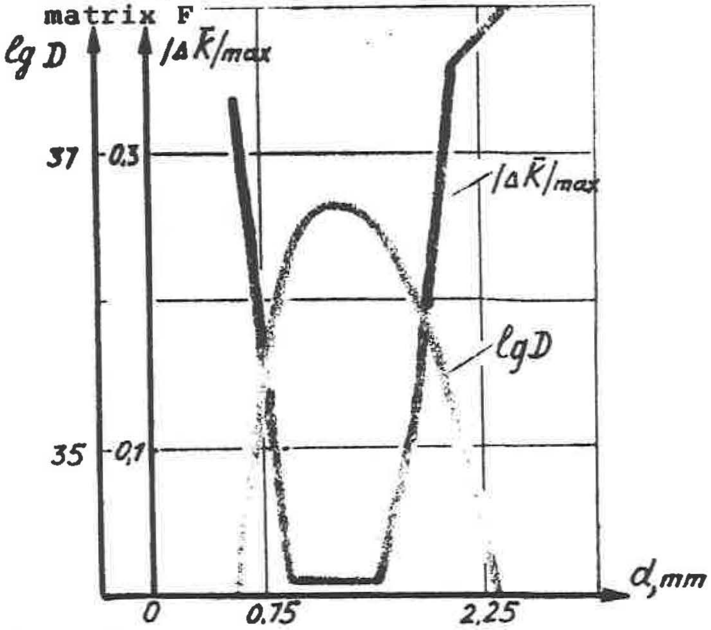
The experiment design is a determination of a number of temperature gauges N, coordinates of their installation into a body $\vec{d} = \{d_i\}_1^N$

D - optimal design: $\vec{\varepsilon} = \vec{\varepsilon}^* = \{N, \vec{d}\}^*: \max\limits_{\vec{\varepsilon}} D[F(\vec{\varepsilon})]$,

where $D[F(\vec{\varepsilon})]$ is a determinant of the corresponding Fisher information matrix F



K(T) reconstruction for D-optimal design



Dependences of criterion value and relative error maximum from d

Mathematical Model № 2
(movable thermosensors and body boundaries)

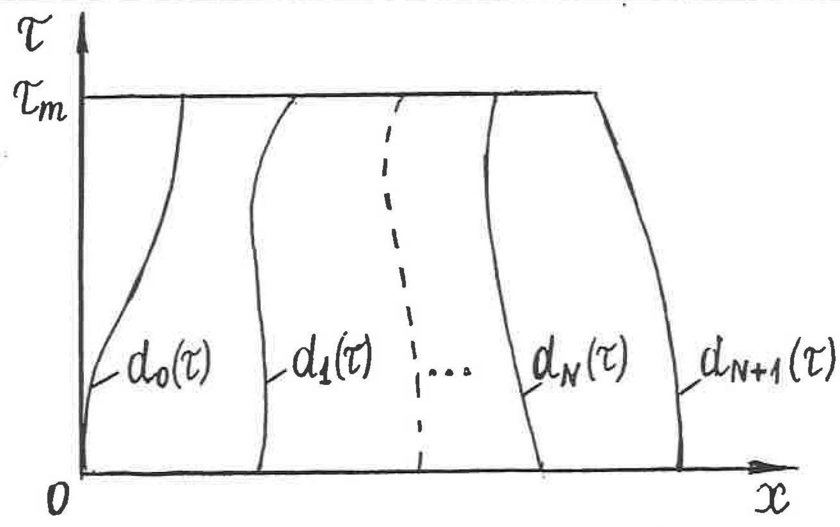
$$C(T)T_\tau = (\lambda(T)T_x)_x, \quad (x,\tau)\in Q_\tau,$$

$$Q_\tau = \{d_0(\tau) < x < d_{N+1}(\tau),\ 0<\tau\le\tau_m\};$$

$$T(x,0) = \xi(x), \quad d_0(0)\le x\le d_{N+1}(0);$$

$$T(d_0(\tau),\tau) = T_0(\tau);\ T(d_{N+1}(\tau),\tau) = T_{N+1}(\tau),$$
$$\tau\in[0,\tau_m];$$

$$T(d_i(\tau),\tau) = f_i(\tau), \quad i=\overline{1,N}, \quad \tau\in[0,\tau_m];$$



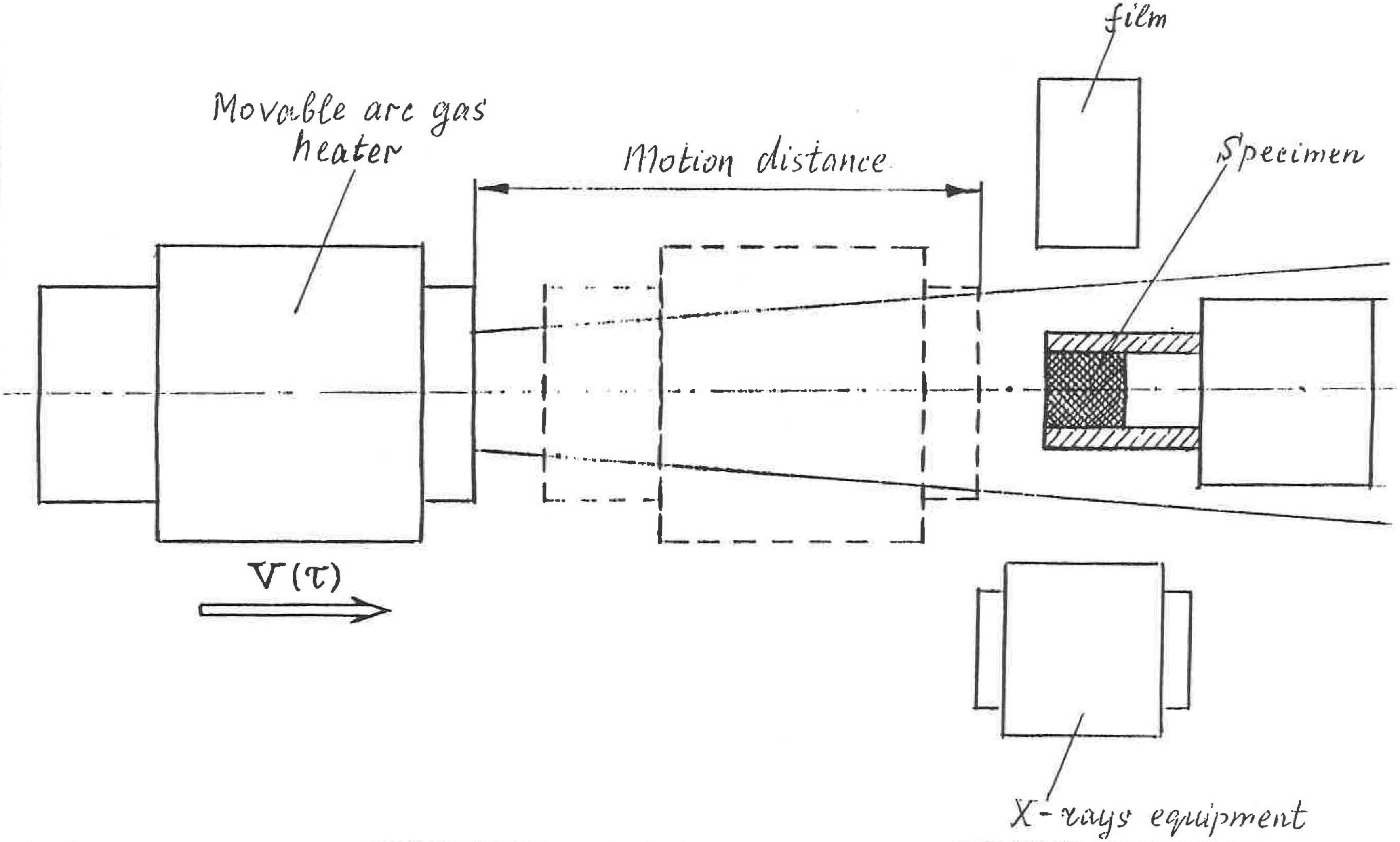
$$d_0(\tau) < d_1(\tau) < \cdots < d_N(\tau) < d_{N+1}(\tau)$$

$$C(T)>0, \quad \lambda(T)>0$$

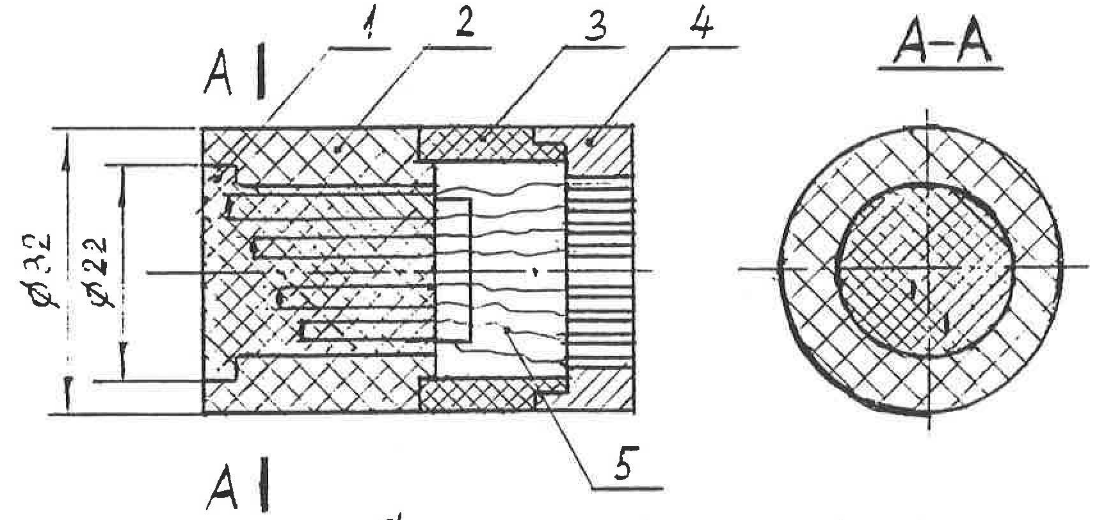
$$\lambda(T):\ \min_{\lambda(T)} |J(\lambda) - \delta_J^2|,$$

$$J(\lambda) = \sum_{i=1}^{N}\int_0^{\tau_m} p_i(\tau)[T(\lambda, d_i(\tau),\tau) - f_i(\tau)]^2 d\tau$$

Experimental Facility

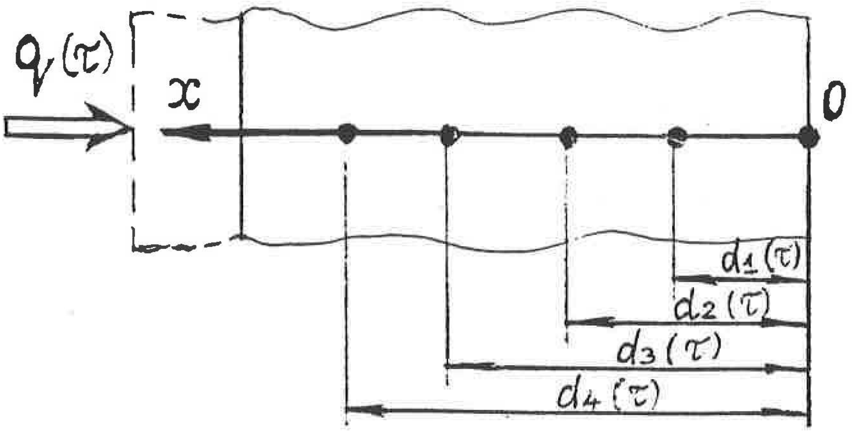


Thermal Properties Investigations of a decomposing material with moving thermocouples



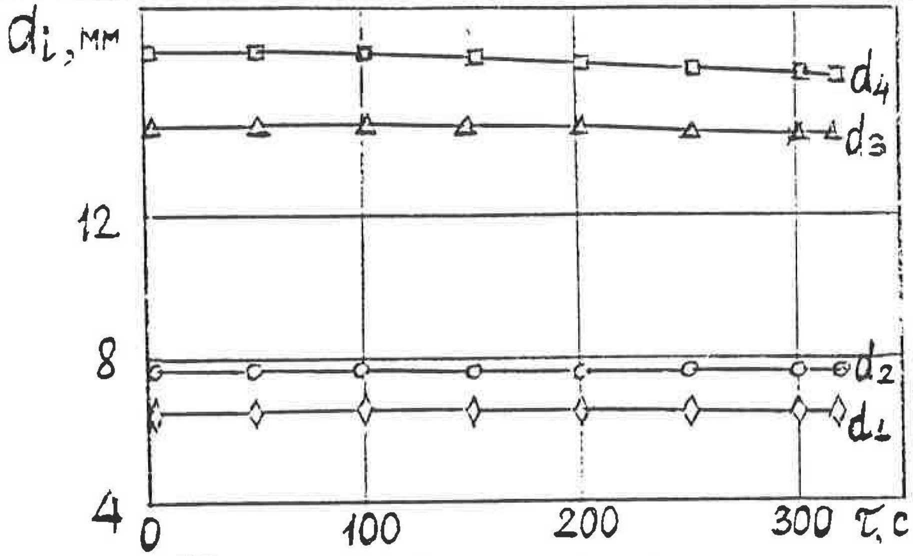
Specimen under investigation

1 - sensing element ; 2,3,4 - frame ;
5 - chromel-alumel thermocouples

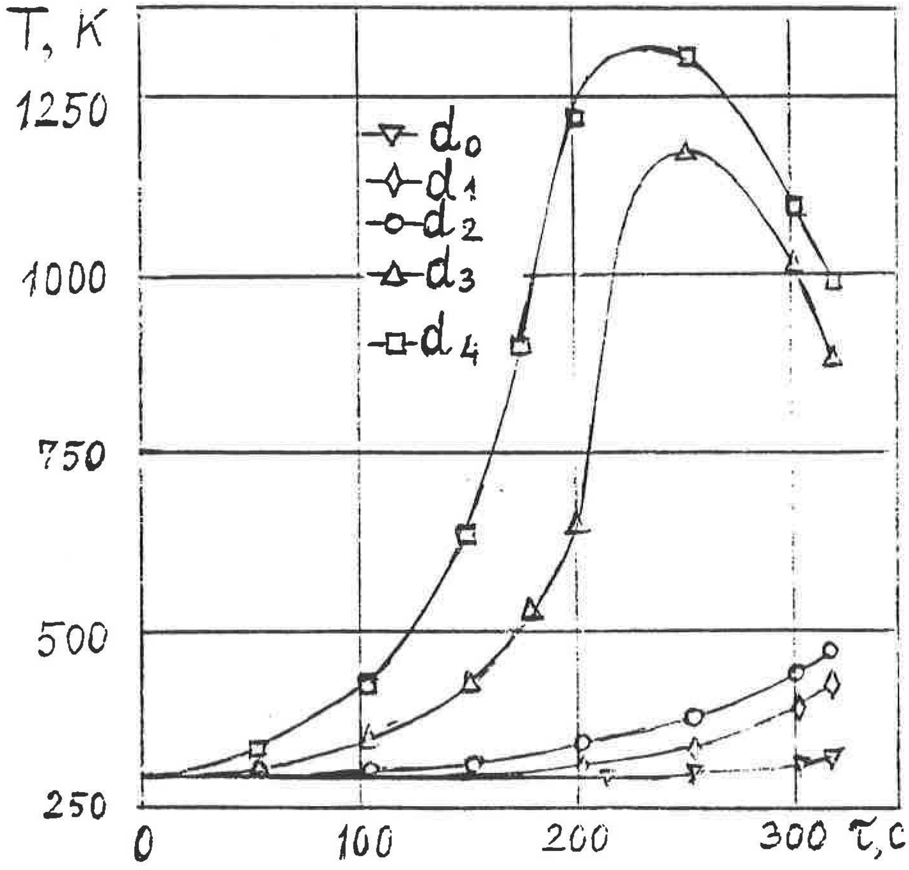


Arrangement of thermocouples in the sensing element

Measurement Results

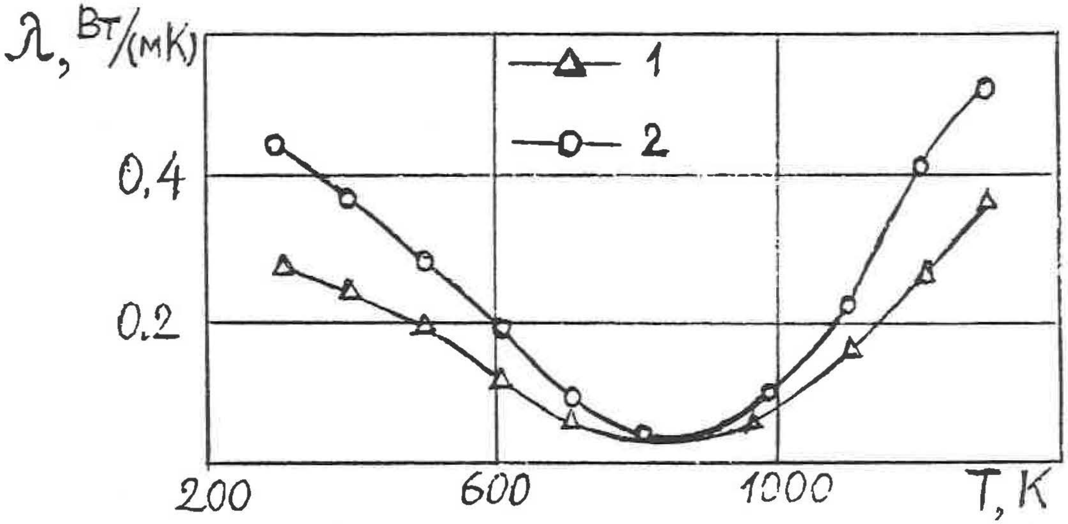


Thermocouple coordinates



Experimental temperatures

Composite thermal protection material **KVT**



Estimation results of thermal conductivity

1 - taking into consideration of the thermocouples displacements;

2 - assuming that thermocouples were fixed

AR4: Study of thermal properties of heat protective materials. Mathematical model of heat-and mass transfer in decomposing materials

$$C(T)\frac{\partial T}{\partial \tau}=\frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right)-P\left(T,\frac{\partial T}{\partial \tau}\right)\frac{\partial T}{\partial x}-S\left(T,\frac{\partial T}{\partial \tau}\right),\quad x\in(0,\delta),\ \tau\in(0,\tau_m];$$

$$T(x,0)=T_0,\quad x\in[0,\delta];\quad T(0,\tau)=f_0(\tau),\quad T(\delta,\tau)=f_N(\tau),\quad \tau\in[0,\tau_m];$$

$$P\left(T,\frac{\partial T}{\partial \tau}\right)=\frac{dh_g(T)}{dT}\int_0^{\delta}\frac{\partial m_g}{\partial x}dx;\qquad S\left(T,\frac{\partial T}{\partial \tau}\right)=h_g(T)\frac{\partial m_g}{\partial x},$$

$$\frac{\partial m_g}{\partial x}=\begin{cases}(1-K_T)\rho_0\left(A+\Delta A\frac{\partial T}{\partial \tau}\right)z^n\exp\left(-E/RT\right), & T>T_r\\ 0, & T\leqslant T_r\end{cases}$$

$$T_r=T_{r_0}+a_1\left(\frac{\partial T}{\partial \tau}\right)^{a_2},\qquad z=(\rho-\rho_0)(\rho_0-\rho_c),$$

where m_g, h_g are specific mass flow rate and enthalpy of gaseous products;

$K_T=\rho_c/\rho_0$ is a limiting value of the coke number;

ρ_0, ρ, ρ_c are initial, current and finite values of the material density;

z is a concentration of a decomposing component;

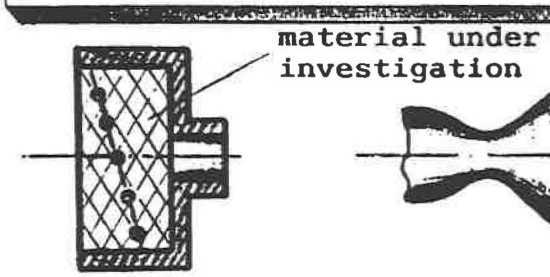
$a_1, a_2, A, \Delta A, n, E$ are thermodestructive parameters;

R is a universal gas constant;

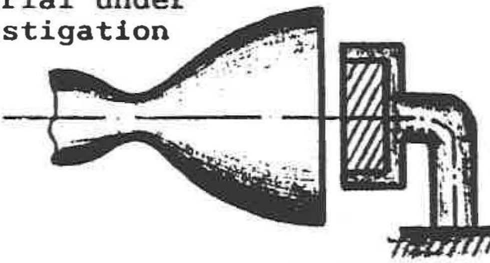
T_{r_0} is a temperature of the start of phisico-chemical conversions in the material

Input data: $T(d_i, \tau) = f_i(\tau)$, $i = \overline{1, N-1}$, $N \geqslant 2$, $0 < d_1 < d_2 < \dots < d_{N-1} < b$

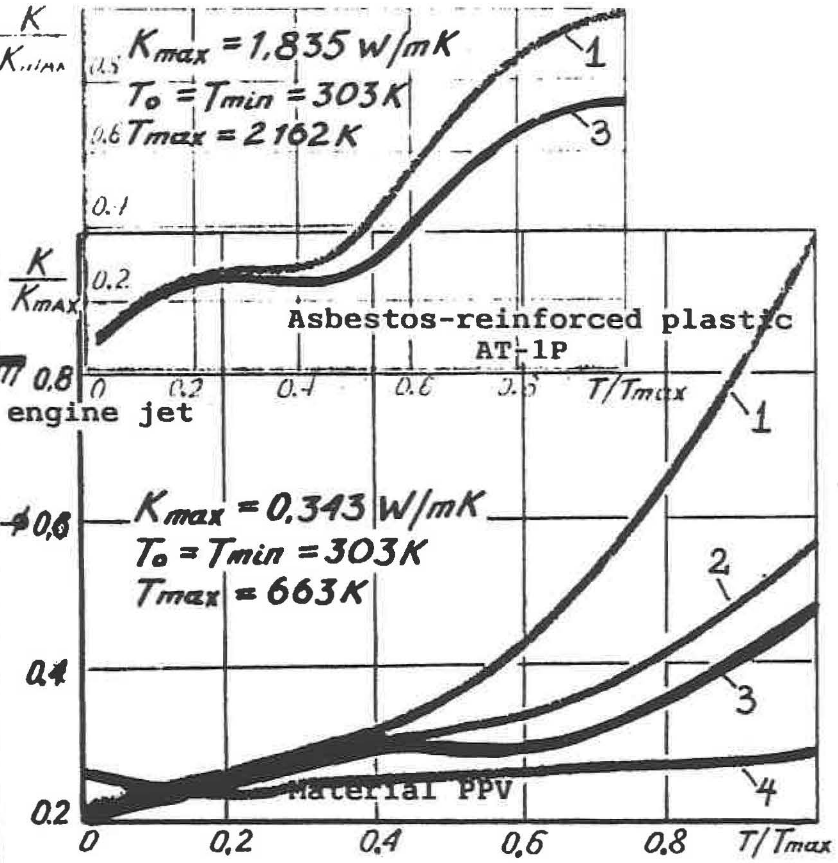
K(T) - is an unknown function

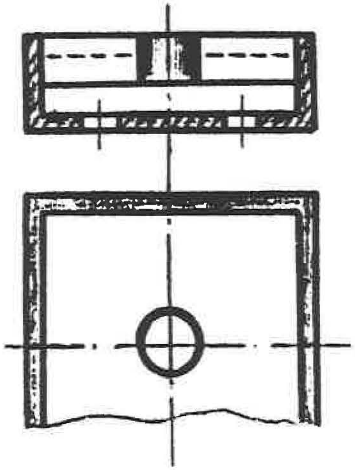


Specimen I

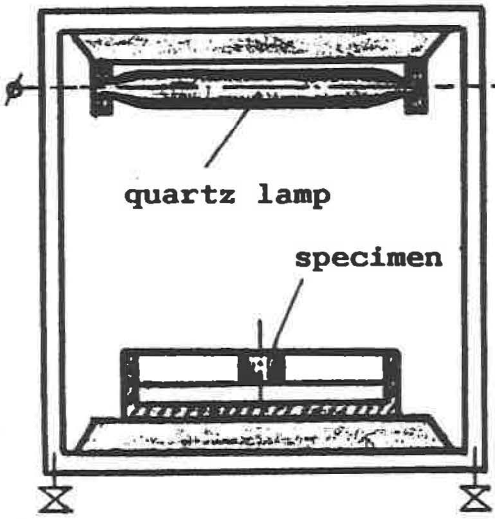


Testing in engine jet





Specimen II



Testing in termal-vaccum chamber

Processing of experimental data:

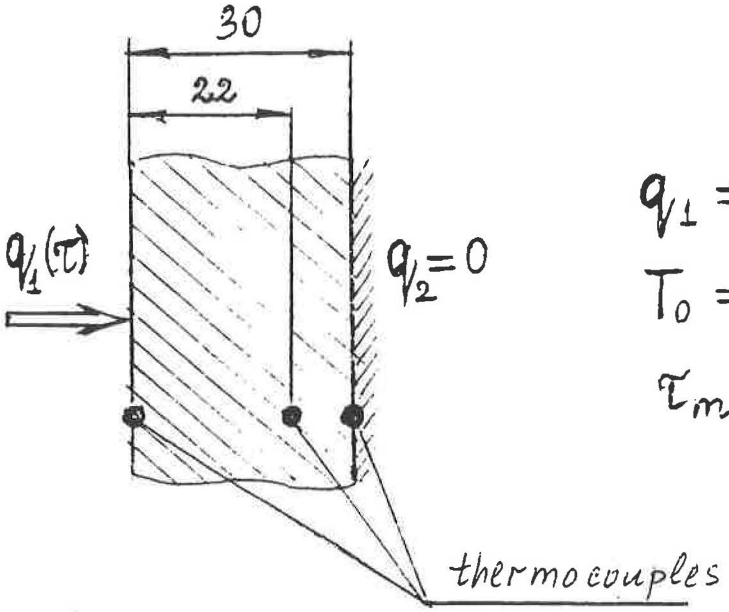
1 - above-stated model

2 - above-stated model at $\Delta A = 0$

3 - uniform heat conduction problem (P=S=0)

4-monotonic heating method

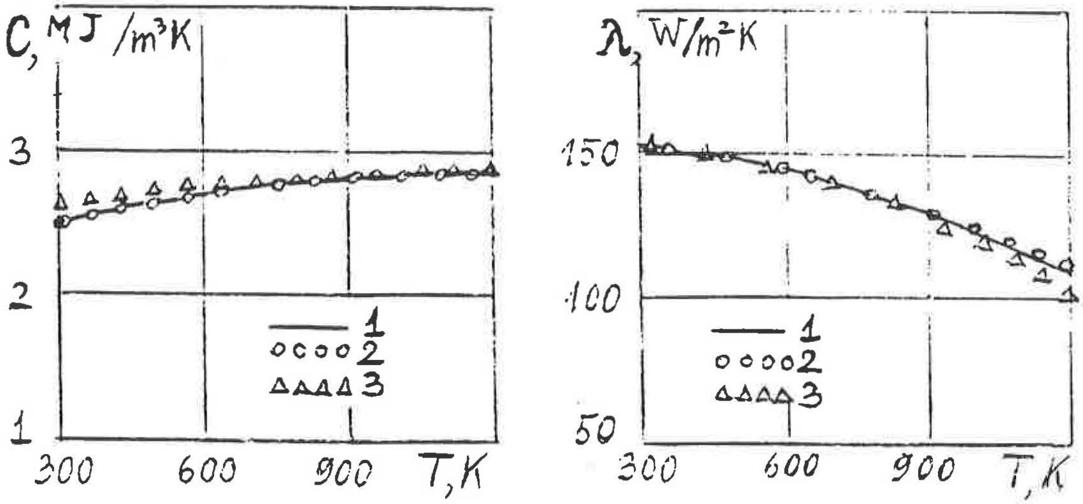
Simultaneous determination of two thermal properties



$q_1 = 190\tau$ kW/m^2,

$T_0 = 300$ K,

$\tau_m = 20$ s



Estimations of volumetric heat capacity C(T) and thermal conductivity λ(T):
1 – exact solutions; 2 – calculations for exact input data; 3 – calculations for perturbated input data, $3\sigma = 0.05 T_{max}$

AR5: Thermal and Gas - dynamic Tests

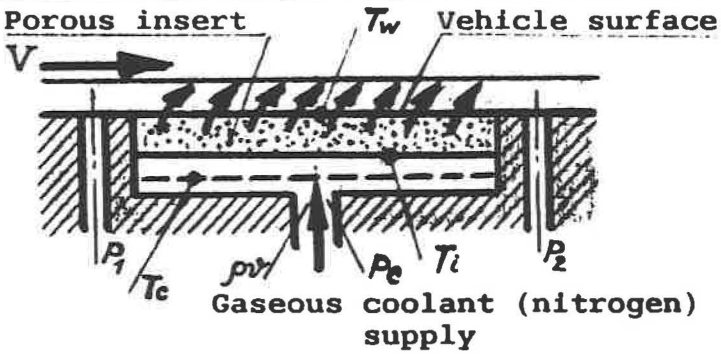
- Diagnostics of heat transfer boundary conditions and heat loads on structures during full-scale testing
- Identification of thermal properties of heat-shield materials

AR6: Thermal - vacuum Test of Spacecrafts

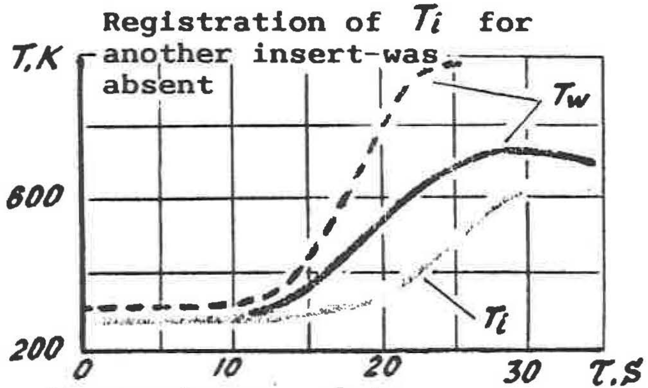
PROCEDURE OF TESTING

1. Special preliminary testing of object for the purpose of identification and correction of mathematical models of heat transfer processes
2. Choice of thermal simulator mode (inverse problem of control type)
3. Regular testing

AR7: Study of Porous Cooling (flight experiment)



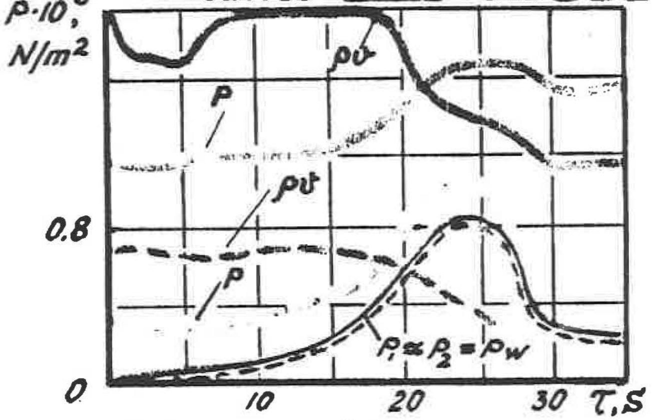
Equipment for investigation of porous cooling



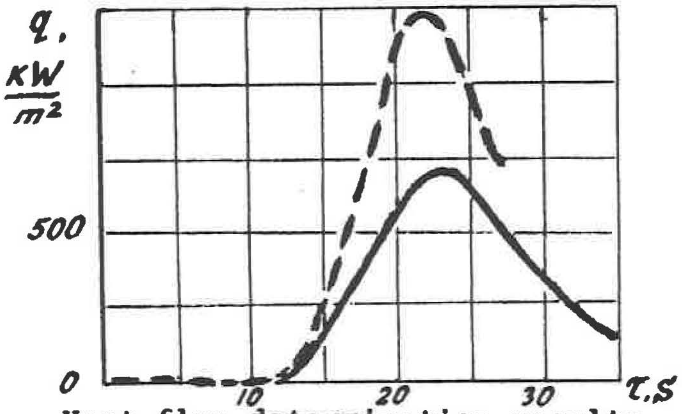
Temperatures data

Pressures before (P_1) and after (P_2) the insert, and pressure of the coolant supply (P_c), temperatures of outer (T_w) and inner (T_i) surfaces, and temperature of supplyed coolant (T_c), and mass flow rate of coolant (ρv) are measured values.

Curves ▬▬ and ‒ ‒ ‒ correspond to two different experimental conditions



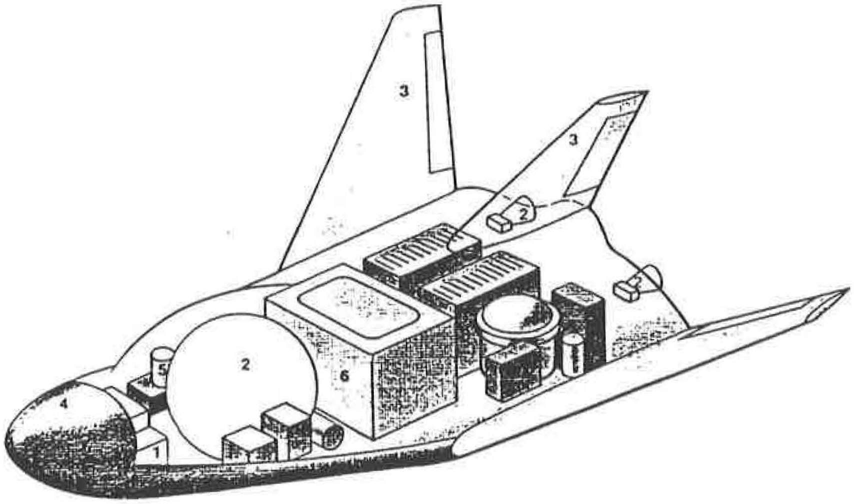
Pressures data



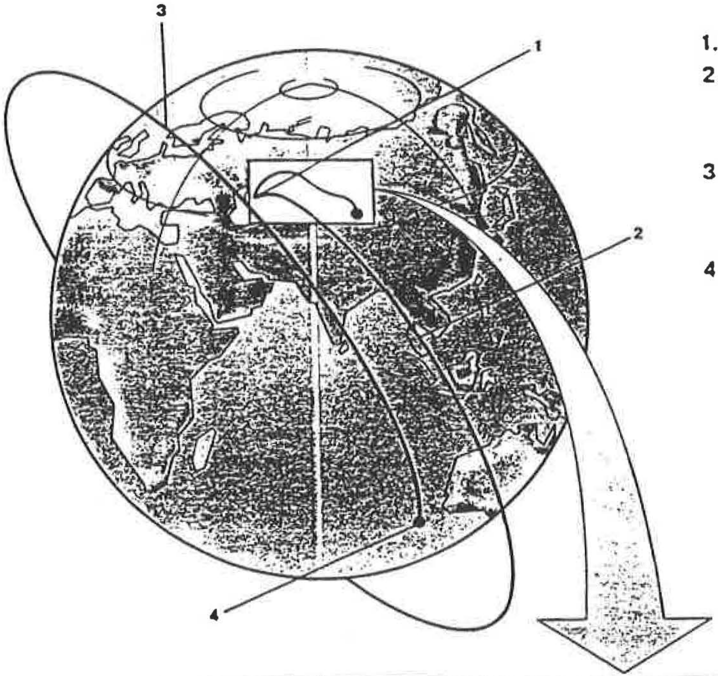
Heat flux determination results

THE FLIGHT-TEST COMPLEX "BOR"

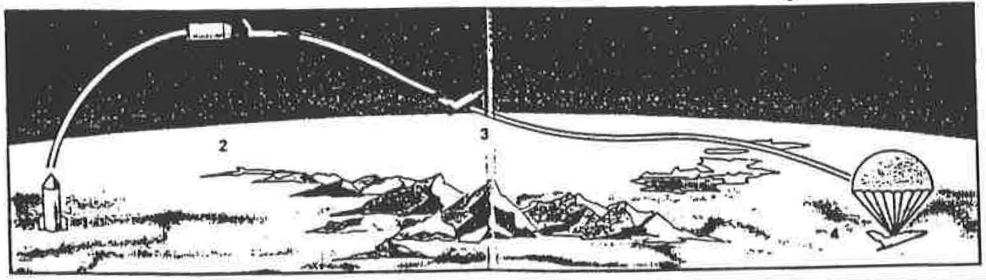
THE COMPLEX FOR FLIGHT RESEARCHES INTO THE PROBLEMS OF AERODYNAMICS, FLIGHT DYNAMICS AND THERMAL PROTECTION OF FUTURE AIR-SPACE PLANES.



1. Power supply source
2. System for gas-dynamic stabilization in space
3. Aerodynamic controls
4. Elements of thermal insulation being tested
5. Research equipment
6. Rescue system
7. Navigation and control system
8. System of radiotelemetry

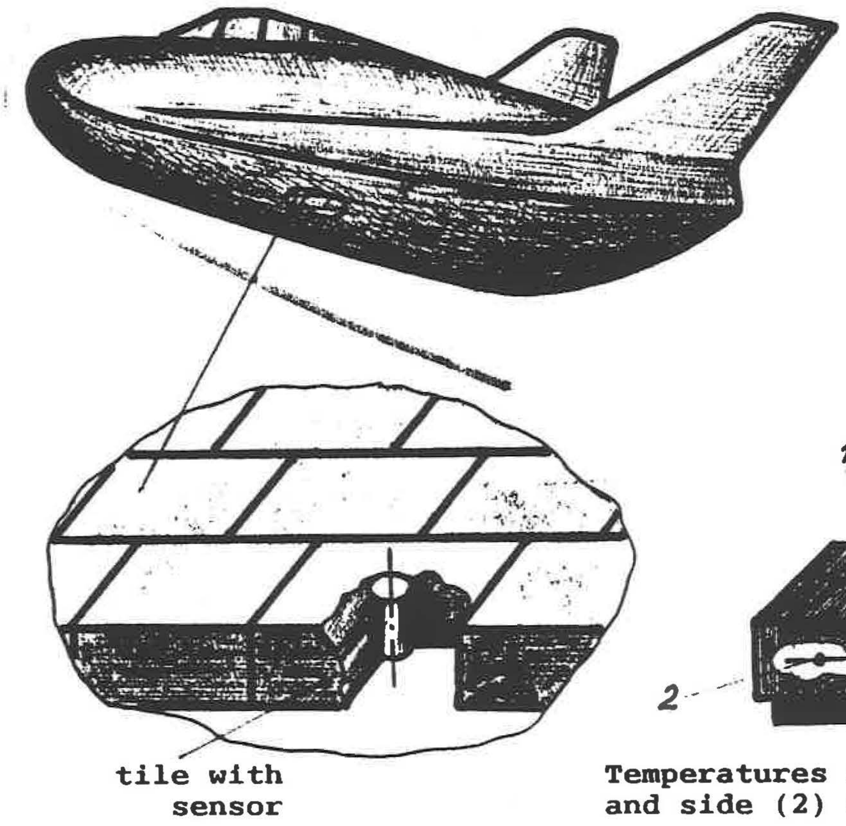


1. Launch
2. Injection into orbit, separation from booster
3. Atmospheric portion of flight trajectory
4. Rescue on parachute

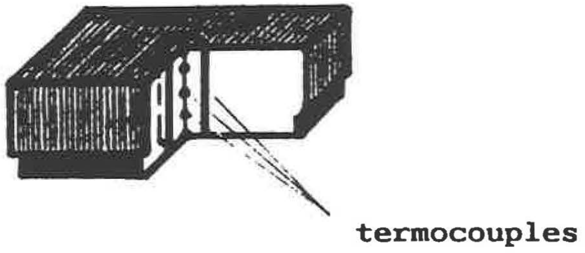


AR 8 : Study of thermal modes at course of flight tests of "Bor-4" automatic re-entry vehicle

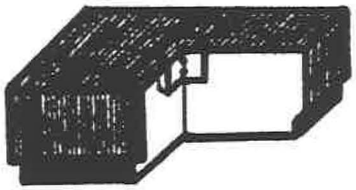
Bor-4



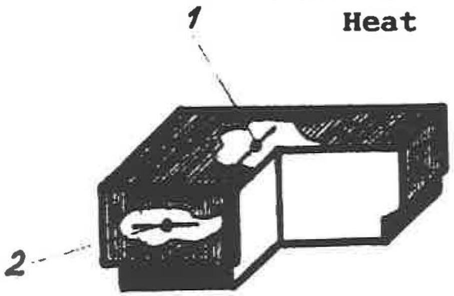
Tiles with different sensors:



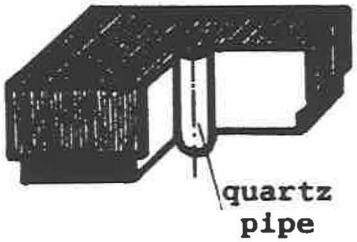
Temperature distribution



Heat flux

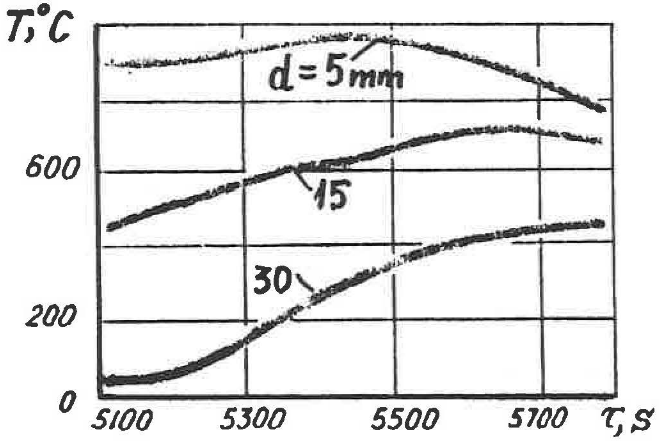


Temperatures of outer (1) and side (2) surfaces

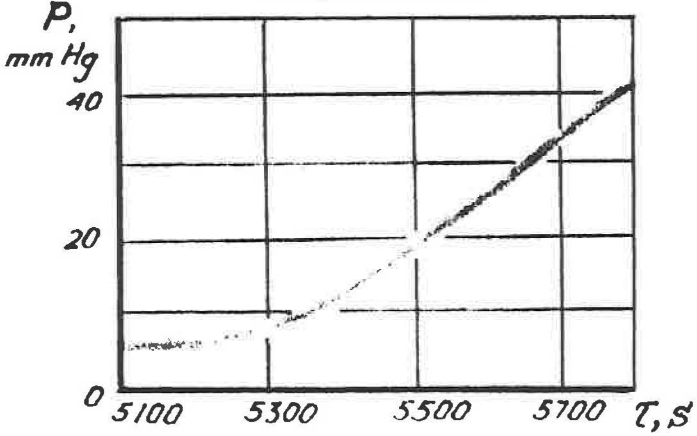


Pressure

Direct measurements :

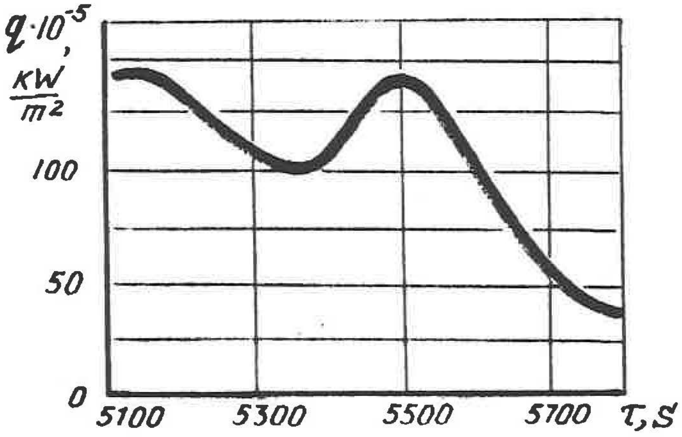


Temperatures

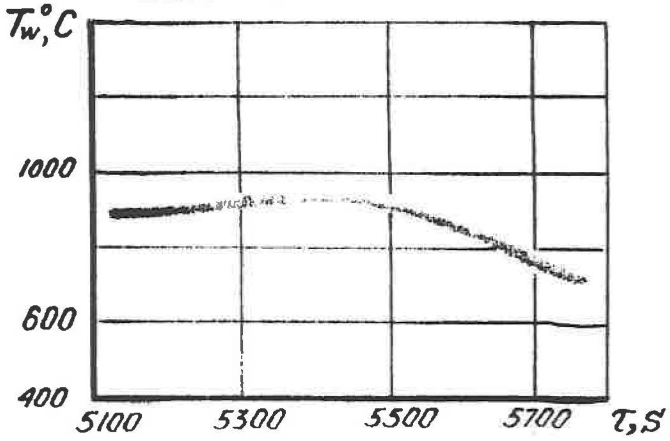


Pressure

Results of solving of I H C P :

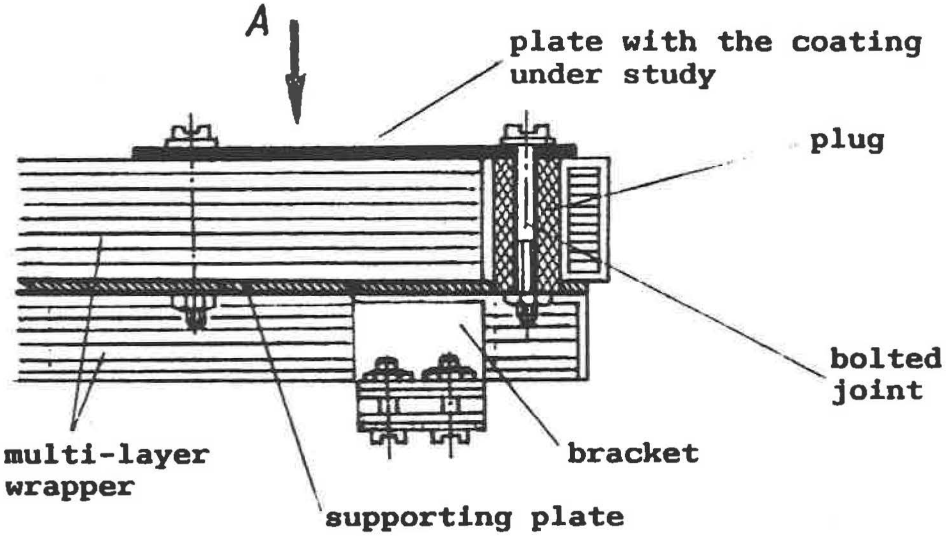


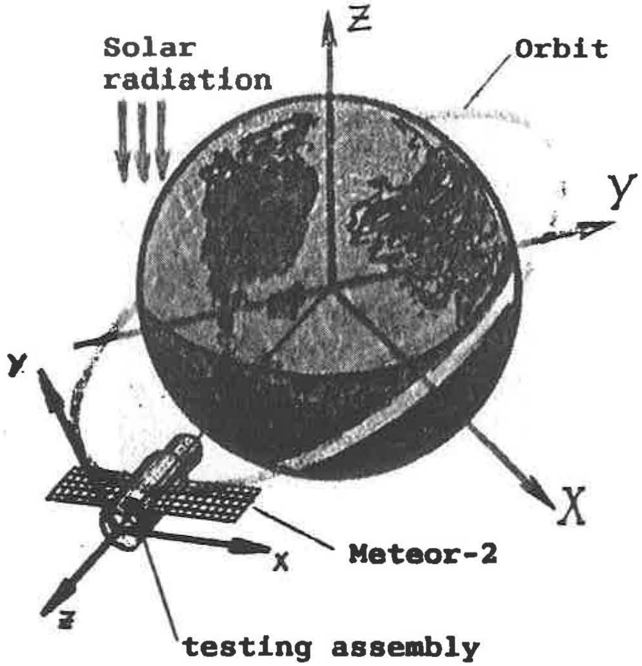
Heat flux

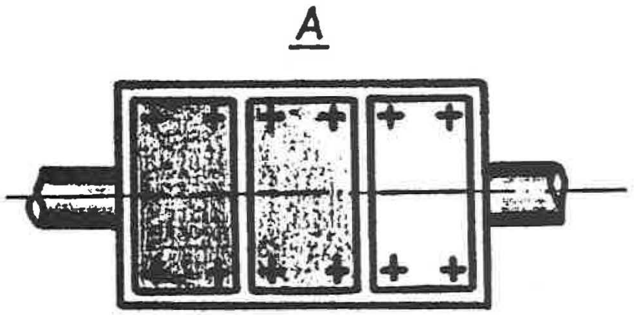


Temperature of surface

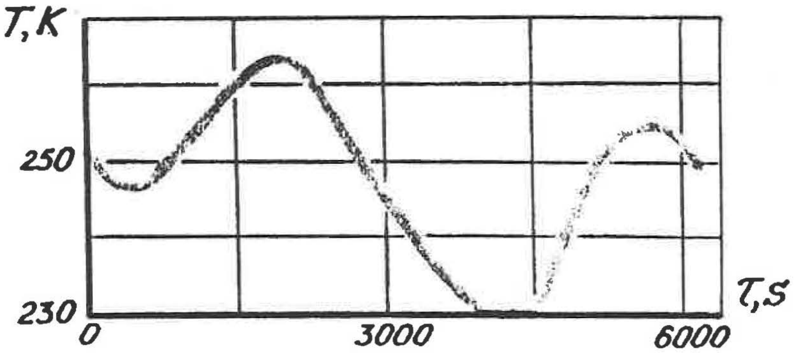
TP 1	AR9: Study of stability of the thermocontrol coating (TCC) of spacecraft at orbit conditions	3/12







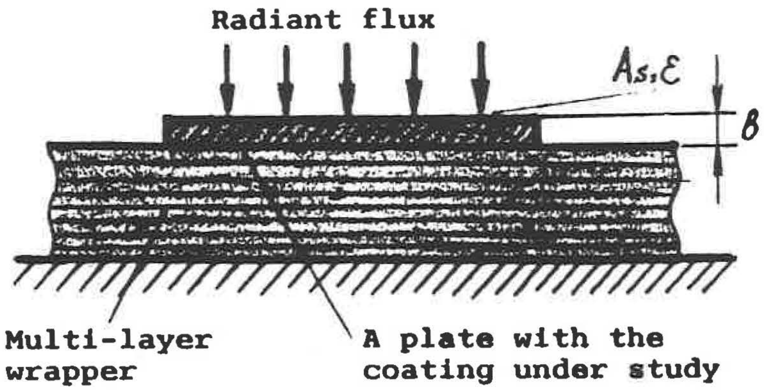
Assembly of the three specimens with different coatings



Type of telemetering data of temperature

TP
1

Example 3: The Diagnostics of radiation characteristics of thermal control coatings



Mounting of a specimen on the surface of a spacecraft

The unknown quantities:

$\boxed{A_s}$ is a solar radiation integral absorptivity factor,

$\boxed{\varepsilon}$ is an integral semi-spherical emissivity

Mathematical model:

$$C_m \frac{dT}{d\tau} = A_s\left[q_s(\tau) + q_R(\tau)\right] + q_\varepsilon(\tau)\varepsilon - \varepsilon\sigma_0 T^4(\tau)$$

$$\tau \in (0, \tau_m), \qquad C_m = C_p \rho \delta \,;$$

$$T(0) = T_0$$

Experimental values of the temperatures:

$$f(\tau_i) = T(\tau_i) + \xi_i \,,$$

$i = \overline{1,N}$, ξ_i are measurement errors

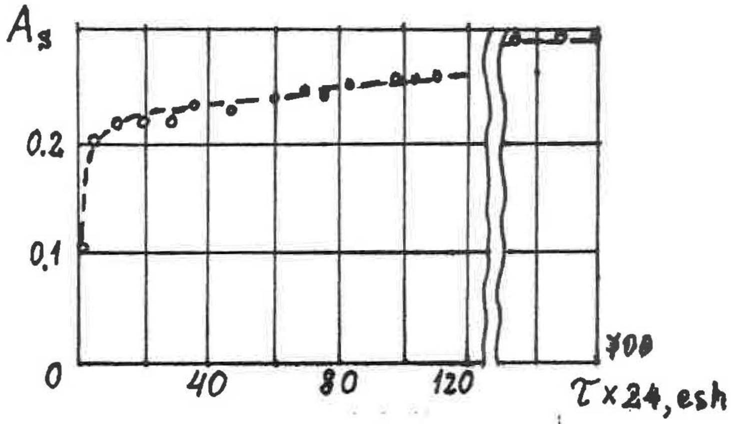
q_s is a direct solar radiation,

q_R is a reflected from the Earth solar radiation,

q_ε is an Earth self-radiation

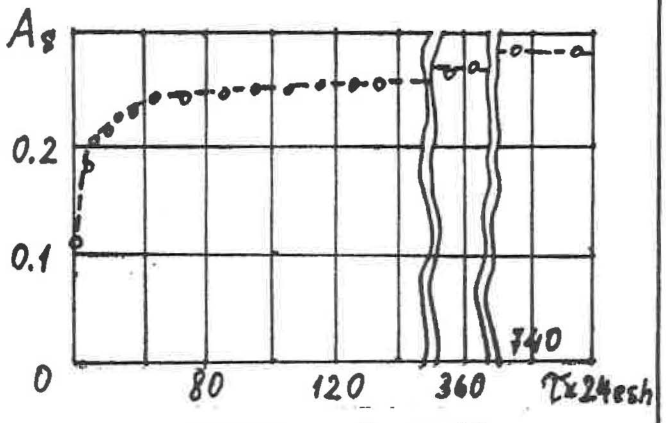
Continuation

Experimental Results



Метео - 2, No4

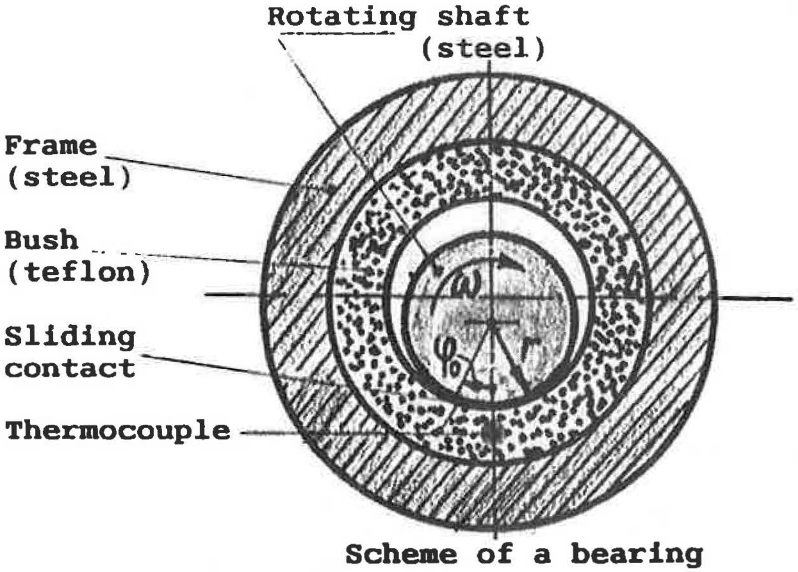
TCC : 050



Метео - 2, No12

TCC : on base of Teflon

AR10: Identification of Heat Loads and Friction Moment of Sliding Bearings



Scheme of a bearing

There is sources of heat in the contact zone ($-\varphi_0 < \varphi < \varphi_0$):

$$Q(\varphi,\tau) \simeq Q(\tau) \quad \text{at} \quad \omega > 5 \text{ rad/s}$$

$$Q(\tau) = 2 r_1 \varphi_0 q(\tau)$$

– is determined from IHTP solution

Spesific rate of heat flow:

$$q(\tau) = k_f p(\tau) \vartheta(\tau),$$

where k_f is a friction coefficient;
p is a pressure in sliding contact;
ϑ is a sliding velocity

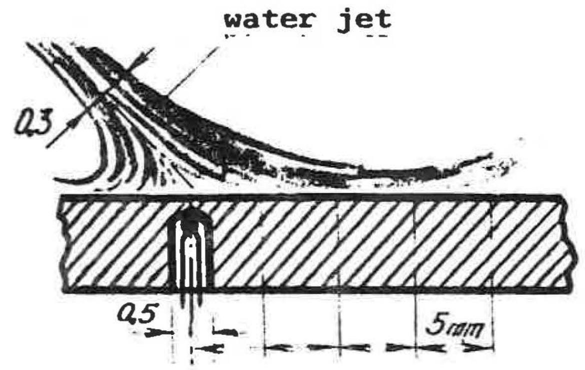
Friction moment:

$$M_f(\tau) = Q(\tau) \cdot r / \vartheta$$

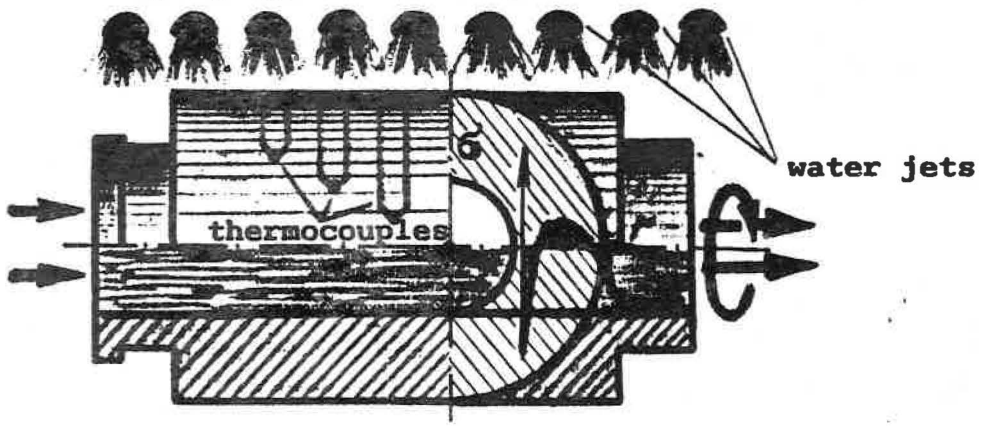
EXPERIMENT: $\vartheta = 0.39$ m/s ; diameters of the bush: $\varnothing 32 \times \varnothing 26$ mm

diameter shaft: $\varnothing$ 25.5 mm

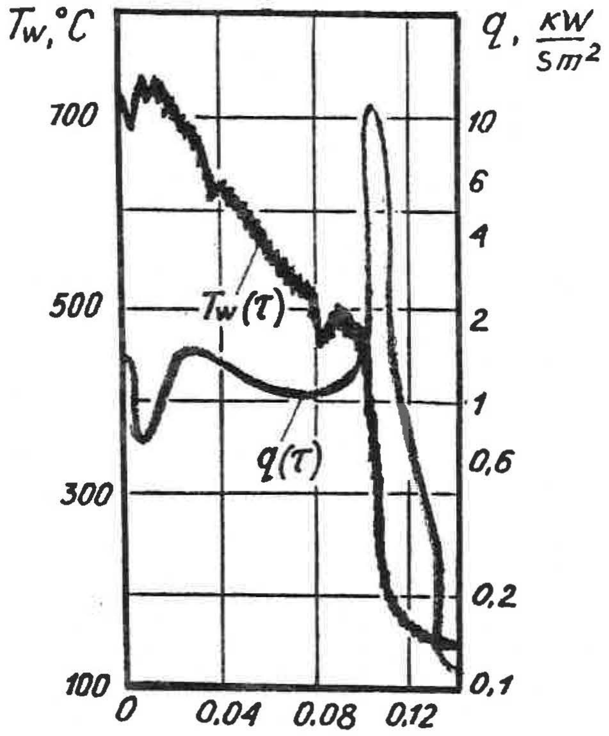
AR 11: An experimental study of the cooling of hot metal with water jets



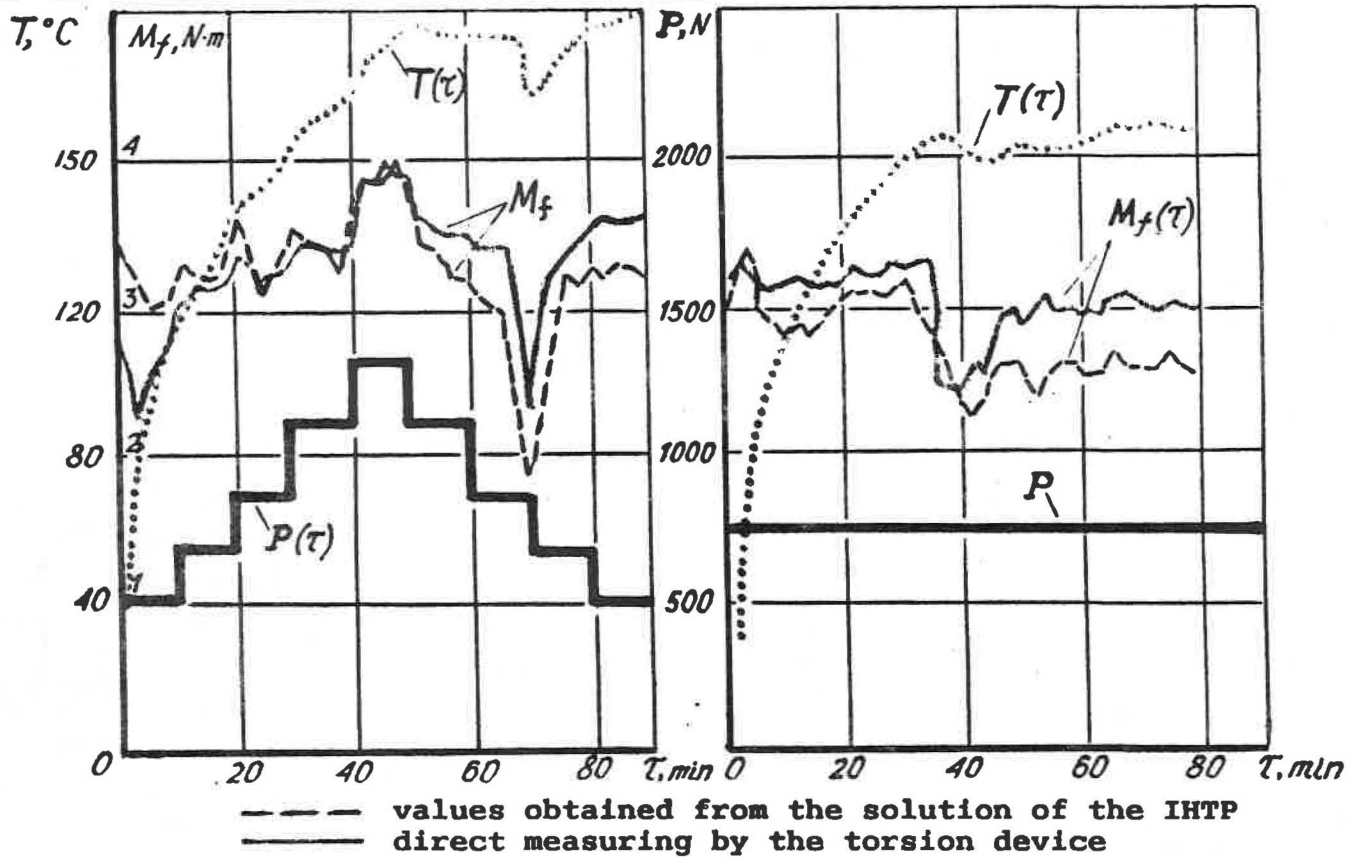
The cooling of high-temperature steel strip equipped with temperature sensors



Thermal treatment of steam turbine rotors



Surface temperature and heat flux at cooling steel strip



Results of determination of the friction moment
(P is a load on rotating shaft)